Asset Pricing with Speculative Trading[†]

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June 1999

Abstract

Speculative trading stems from disagreements among traders. Besides the approaches based on the existence of private information (and noise traders) or the differences of opinions, Harrison and Kreps(1978) and Morris(1996) relied on the presence of diverse beliefs to explain speculative phenomena. This paper proposes a new model of speculative trading by introducing rational beliefs of Kurz(1994) and Kurz and Wu(1996). Agents hold diverse beliefs which are "rational" in the sense of being compatible with observed data. In a non-stationary environment the agents may learn only about the stationary measure of observed data. Agents' beliefs can be non-stationary and diverse even when their stationary measures become the same as that of the data with complete learning. In a Markovian framework of dividends and beliefs, we obtain analytical results on how the speculative premium depends on the extent of heterogeneity of beliefs. In addition, we demonstrate the possible emergence of endogenous uncertainty (as defined by Kurz and Wu(1996)) and the persistent presence of diverse beliefs and positive speculative premiums.

[†]This is substantially revised from our paper previously circulated under the title "Speculation with Rational Beliefs". We are grateful to Mordecai Kurz, Kenneth Arrow, Peter Hammond, Michael Magill, Carsten Nielsen, Martine Quinzii, and Fernando Zapatero for helpful comments. We are also grateful to Academia Sinica and the National Science Council of R.O.C. for their indispensable support.

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1. Introduction

Speculation has been a major research topic for economists, especially in light of recent financial crises and speculative attacks on currency and stock markets. Faced with a similarly turbulent world sixty years ago, Lord Keynes brought to people's attention the relationship between speculation and subjective expectations and compared the determination of stock prices to a "beauty contest." Investors are "concerned, not with what an investment is really worth to a man who buys it for keep, but with what the market will value it at, under the mass psychology, three months or a year hence."(Keynes(1936)). According to Kaldor(1939), speculation "may be defined as the purchase (or sale) of goods with a view to resale (or repurchase) at a later date ...." Such kind of speculative behavior cannot exist in a world of complete markets or rational expectations (see Arrow(1953) and Tirole(1982)), where investors do not change their asset holding even when markets reopen later. So the appropriate framework to study speculative trading is the one of incomplete financial markets with sequential trading. Speculative trading can then stem from disagreements among investors. The purpose of this paper is to probe further into the relationship between speculation and subjective valuation and provide a rigorous foundation for a theory of asset pricing with speculative trading.

There are at least three approaches for modeling disagreements and speculation. The first is a large literature based on the presence of private information and noise (liquidity) investors(see, for example, Grossman and Stiglitz(1980) and DeLong et al.(1990).). Then the difference-of-opinion approach by Varian(1985,1989) and Harris and Raviv(1993) dispenses with the noise investors and obtains diverse posterior beliefs from the differences in the way investors interpret common information. There is also a third method to explain diverse posterior beliefs by relaxing the assumption of common prior, as in Harrison and Kreps(1978) and Morris(1996). While Harrison and Kreps studied an economy with dividends distributed as a Markovian Chain, Morris adopted a simplified framework of independently and identically distributed dividends and demonstrated the presence of speculative premiums. According to Morris, the result of Harrison and Kreps "has apparently been largely ignored, presumably because of the assumption of (unmodeled) heterogeneity of expectations." (Morris(1996), p.1112). We intend to correct this weakness by providing a theory to justify the sustained presence of diverse beliefs. Our theory is different from Morris(1996) where the heterogeneity of beliefs was present only for a short while. In Morris(1996) the price of a risky asset can be greater than its fundamental value, but only initially, and the difference will converge to zero as investors' beliefs converge over time with Bayesian learning. Such a framework may be more appropriate for modeling asset pricing during initial public offerings, but not for other speculative phenomena.

This paper proposes a new framework to study asset pricing with speculative trading by introducing rational beliefs of Kurz(1994) and Kurz and Wu(1996) (also see Kurz and Schneider(1996) and Kurz and Beltratti(1997)). The theory of rational beliefs assumes that agents have ample data and that an empirical distribution exists and is commonly known to all agents. The theory then shows that the empirical distribution can uniquely be extended to a probability measure on infinite sequences of observations and relative to that measure the process of observed variables is stationary. We call that probability measure "the stationary measure" of the dynamics. The stationary measure may not be the same as the non-stationary measure under which the data was generated to begin with. However, the stationary measure is the common empirical knowledge on which all the agents agree.

Investors have diverse beliefs which are "rational" in the sense of being compatible with observed data. In a non-stationary environment the investors can learn only about the stationary measure of observed data. Although the stationary measures of investors' beliefs will become the same as that of the data with complete learning, these beliefs may stay non-stationary and diverse. For example, they can choose from a set of rational beliefs compatible with data(having the same stationary measure), but the timing of such choices may be non-stationary and different. Therefore, investors may disagree even when they are allowed to learn with a large number of observations. Unlike Morris(1996), we adopt the Markovian framework of Harrison and Kreps(1978) to model dividends and agents' beliefs. In Morris(1996) posterior beliefs must converge, but our framework allows the investors' beliefs to stay diverse even with complete learning. Our framework provides a foundation for the continued presence of heterogeneous expectations in speculative trading, which was not offered by Harrison and Kreps. The framework of rational beliefs also enables us to study many interesting phenomena. In particular, we demonstrate the emergence of endogenous uncertainty (see Kurz and Wu(1996) and Huang and Wu(1999)) and the continued deviation of asset prices from agents' valuation if obliged to hold the asset forever. We also show that positive speculative premiums will persist in a Markovian model of speculative trading.

Before introducing rational beliefs in Section 4, we will discuss the basic model in section 2 and analyze the properties of asset prices with a general Markovian belief system in Section 3. We demonstrate that the equilibrium asset prices can be determined on the basis of a "representative belief", which is constructed systematically from heterogeneous beliefs of agents in the economy. The equilibrium asset prices with speculative trading were demonstrated to be no less than any investor's valuation by Harrison and Kreps(1978). Morris(1996) showed that asset prices are strictly greater than any investor's valuation under certain conditions, but in a simplified framework of i.i.d. binomial distribution for dividends. In our framework of Markovian dividend processes and Markovian beliefs, we find the conditions for the emergence of positive speculative premiums by utilizing the technique of constructing a "representative belief" for the economy. In addition, we provide analytical results on how the premium depends on the extent of heterogeneity of beliefs while Morris(1996) obtained results only from numerical simulation. Section 5 concludes.

2. The Basic Model

We consider an economy with a finite number of types of investors ($i = 1, \cdots, K$), each type having different expectations about the future values of a risky asset. Following Harrison and Kreps(1978) and Morris(1996), we also assume that investors are risk neutral, that each type of investors has infinite collective wealth and that all investors cannot sell the asset short. As discussed by Harrison and Kreps(1978, Section VI), such a model is a good approximation to a world of risk averse investors with finite wealth. As for the no short sales assumption, allowing some finite amount of short selling would not change the main results (see Morris(1996), p.1122) in this type of models.

All investors have access to the same information set and future dividends $\{d_t\}$ of the risky asset are believed to follow a specific exogenous stochastic process. Harrison and Kreps(1978) allowed for a general functional dependence of dividends on the current information set while

Morris(1996) considered the case of an i.i.d. dividend process. In our model we assume that the dividends follow a finite-state stationary Markov chain. Suppose the transition matrix of investors' beliefs is an $S \times S$ matrix,

$$Q^i = \begin{bmatrix} q^i_{11} & q^i_{12} & \cdots & q^i_{1S} \\ \vdots & \vdots & \cdots & \vdots \\ q^i_{s1} & q^i_{s2} & \cdots & q^i_{sS} \\ \vdots & \vdots & \cdots & \vdots \\ q^i_{S1} & q^i_{S2} & \cdots & q^i_{SS} \end{bmatrix} = \begin{bmatrix} q^i_1 \\ \vdots \\ q^i_s \\ \vdots \\ q^i_S \end{bmatrix}, \tag{1}$$

where q^i_s is the sth row vector of Q^i and $q^i_{ss'}$ represents the probability of state s' occurring in the next period given that the current state is s, $s = 1, \cdots, S$. The elements of q^i_s should be between 0 and 1 and the sum of all elements is equal to one. So the investors' belief about the dividends also follows a Markov chain. As in Kaldor or Harrison and Kreps, we say that investors exhibit "speculative trading behavior" if they are willing to pay more for the risky asset with a right to resell than what they would pay if obliged to hold it forever. Investors are willing to pay a "speculative premium" for the anticipated gains from speculative trading.

Before considering the general $S \times S$ case in the next section, we will illustrate some basic ideas in the framework of 2×2 Markov chains ($S = 2$). Suppose the belief of investors of type i is

$$Q^i = \begin{bmatrix} 1 - a^i & a^i \\ 1 - b^i & b^i \end{bmatrix}, \; i = 1, \cdots, K\,, \tag{2}$$

where a^i, b^i are all between 0 and 1. Suppose there are two possible values of dividends

$$\vec{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},$$

with $d_2 > d_1$. Let γ denote the common discount rate, $\gamma < 1$, and $\vec{p^i}$ be the expected present value of subjectively evaluated dividends to an investor of type i. With the risk neutrality assumption we can derive the "subjective valuation" $\vec{p^i}$ of an investor of type i if obliged to hold the asset forever:

$$\vec{p^i} = \gamma Q^i (\vec{p^i} + \vec{d})\,. \tag{3}$$

By using the following lemma, we can solve for $\vec{p^i}$ uniquely:

$$\vec{p^i} = (I - \gamma Q^i)^{-1}\gamma Q^i \vec{d}. \tag{4}$$

Lemma 1: For a 2×2 transition matrix Q^i, $I - \gamma Q^i$ is invertible.

pf: By simple calculation, $\mathrm{Det}(I - \gamma Q^i) = (1-\gamma)(1-(a^i - b^i)\gamma) > 0$ with $\gamma < 1$. $Q.E.D.$

When there is only one type of investors in the economy or when there is no disagreements among different types of investors ($Q^i = Q$, $\forall i$), the market equilibrium price can be represented by (4). However, when we have heterogeneous beliefs, the phenomenon of speculative trading occurs as shown in the following example.

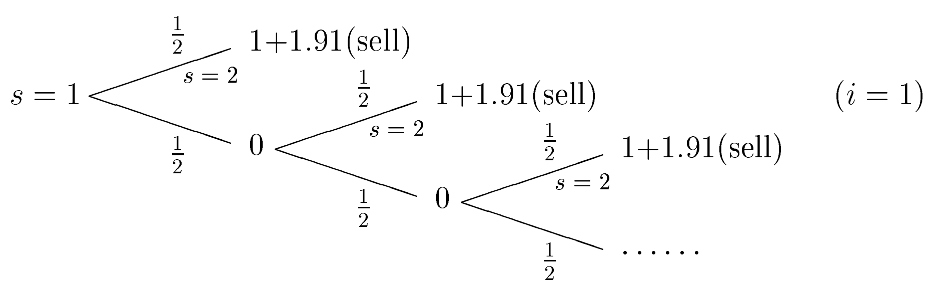
Example 1: Following the example of Harrison and Kreps(1978), we assume $K = 2$, $\gamma = 0.75$, $d_1 = 0$, $d_2 = 1$,

$$Q^1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \ Q^2 = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

By applying equation (4) we can find

$$\vec{p^1} = \begin{bmatrix} \frac{4}{3} \\ \frac{11}{9} \end{bmatrix} = \begin{bmatrix} 1.33 \\ 1.22 \end{bmatrix}, \ \vec{p^2} = \begin{bmatrix} \frac{16}{11} \\ \frac{21}{11} \end{bmatrix} = \begin{bmatrix} 1.45 \\ 1.91 \end{bmatrix}.$$

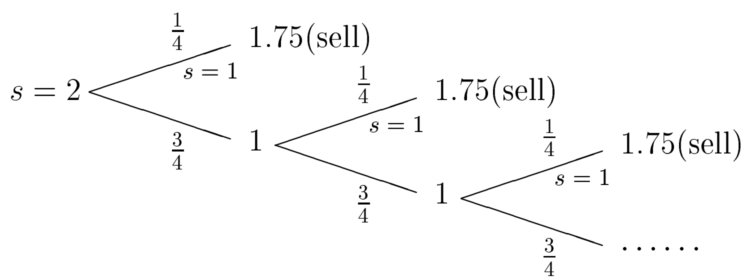
With the infinite wealth and no short sale assumptions, it might be conjectured that investors of type 2 will hold the asset and the market price will be $\vec{p^2}$. However, investors of type 1 can "speculate" by buying the asset in $s = 1$ with the intention to sell it when $s = 2$ occurs, This strategy is shown as in the following graph:



Such a speculative plan can generate a revenue of

$$[\frac{1}{2}(0.75) + (\frac{1}{2})^2(0.75)^2 + \cdots] \cdot (1 + 1.91) = 1.75\,,$$

which is greater than the purchase cost of 1.45 in $s=1$. So the market price in $s=1$ should be at least 1.75. That is, the market price should become $\begin{bmatrix} 1.75 \\ 1.91 \end{bmatrix}$, higher than $\begin{bmatrix} 1.45 \\ 1.91 \end{bmatrix}$, due to the speculative behavior of type 1 investors. Then 1.91 cannot be the market price in $s=2$ since investors of type 2 can "speculate" by buying the asset in $s=2$ with the intention to sell it when $s=1$ occurs, as in the following graph:



$(i=2)$

Such a speculative plan can generate a revenue of

$$[\frac{3}{4}(0.75)+(\frac{3}{4})^2(0.75)^2+\cdots]\cdot 1+[\frac{1}{4}(0.75)+(\frac{3}{4})(\frac{1}{4})(0.75)^2+(\frac{3}{4})^2(\frac{1}{4})(0.75)^3]\cdot 1.75=2.03\,,$$

which is greater than the purchase cost of 1.91 in $s=2$. So the market price in $s=2$ should be at least 2.03. That is, the market price should become $\begin{bmatrix} 1.75 \\ 2.03 \end{bmatrix}$, higher than $\begin{bmatrix} 1.75 \\ 1.91 \end{bmatrix}$, due to the speculative behavior of type 2 investors. Then there will exist another speculation plan and so on. This speculation process will continue until it converges to $\vec{p^*}=\begin{bmatrix} 1.85 \\ 2.08 \end{bmatrix}$.

Harrison and Kreps demonstrate that the infinite progression as in the above example finally stops and achieves a "minimal consistent price scheme", which is also the market price. In the following proposition, we can obtain a characterization of the market price without relying on the limit argument of Harrison and Kreps. We introduce a concept of "representative belief" which is constructed from the current beliefs of all investors. Then the market price can be determined on the basis of the "representative beliefs".

Proposition 1: For the 2×2 case, there exists a unique (stationary) market price $\vec{p^*}$ and a representative belief Q^*, such that

$$\vec{p^*}=(I-\gamma Q^*)^{-1}\gamma Q^*\vec{d}, \tag{5}$$

$$Q^* = \begin{bmatrix} 1 - \max_i a^i & \max_i a^i \\ 1 - \max_i b^i & \max_i b^i \end{bmatrix} . \tag{6}$$

pf: At market equilibrium, for any state s there always exist a type of investors who have the highest subjective valuation for the asset and get hold of it:

$$p_s^* = \max_i \gamma q_s^i(\vec{p^*} + \vec{d}) \, , \; s = 1, 2 \, , \text{ or}$$

$$\vec{p^*} = \max_i \gamma Q^i(\vec{p^*} + \vec{d}) \; . \tag{7}$$

The stationary market equilibrium price p_s^* should satisfy

$$p_s^* = \gamma q_s^{i(s)}(\vec{p^*} + \vec{d})$$

for some type of investors $i(s)$. From the above equation, we can obtain the following equality,

$$p_2^* - p_1^* = \frac{\gamma(b^{i(2)} - a^{i(1)})}{1 - \gamma(b^{i(2)} - a^{i(1)})}(d_2 - d_1).$$

Since $d_2 > d_1$, we have $p_2^* - p_1^* + d_2 - d_1 > 0$. Suppose state 1 occurs, the difference of willingness to pay between investors of type i and j is computed from (7):

$$\gamma q_1^i(\vec{p^*} + \vec{d}) - \gamma q_1^j(\vec{p^*} + \vec{d}) = \gamma(a^i - a^j)((p_2^* - p_1^*) + (d_2 - d_1)),$$

which is positive if $a^i > a^j$. So the investors with maximal a^i will have the highest valuation of the asset when state 1 occurs. Similarly, when state 2 occurs the investors with maximal b^i will have the highest valuation. Substituting these back into equation (7) and applying Lemma 1, we can derive equations (5) and (6). *Q.E.D.*

Next we consider whether there would exist positive speculative premiums in this economy. Speculative premiums are defined to be the differences between the market price $\vec{p^*}$ and the subjective valuation $\vec{p^i}$ by investors of type i ($i = 1, \cdots, K$) if investors are obliged to hold the asset forever. In Example 1, the speculative premiums are positive: $\vec{p^*} - \vec{p^1} > 0$, $\vec{p^*} - \vec{p^2} > 0$. However, if there is an "absolutely optimistic" investor j who has the highest valuation in all states, i.e., $a^j = \max_i a^i$ and $b^j = \max_i b^i$, this investor becomes the representative investor in the market with his belief Q^j being related as Q^*. Then the speculative premium is zero. Only when the market price is strictly greater than the subjective valuation of all investors, we say that there exist positive speculative premiums. Morris(1996) demonstrated the existence of positive risk premiums with an i.i.d. dividend process. In the following proposition, we provide conditions for the existence of positive risk premiums in a Markovian framework, which is the same as the one adopted by Harrison and Kreps(1978). We also characterize how the size of premiums is related to the extent of heterogeneity of beliefs. Without loss of generality, we assume in the proposition that $a^1 = \max_i a^i$, and $b^2 = \max_i b^i$.

Proposition 2: Suppose there is no absolutely optimistic investor in the market for the $S = 2$ case and $a^1 = \max_i a^i$, and $b^2 = \max_i b^i$. Then there exist strictly positive speculative premiums in each state if $a^1 \neq 0$ and $b^2 \neq 1$. The size of speculative premiums is an increasing function of the difference of dividends in the two states $(d_2 - d_1)$ and the extent of heterogeneity of beliefs $(a^1 - a^2$ or $b^2 - b^1)$, that is,

$$\vec{p^*} - \vec{p^1} = \frac{\gamma}{1-\gamma} \cdot \frac{1}{1+(a^1-b^2)\gamma} \cdot \frac{1}{1+(a^1-b^1)\gamma} \cdot (d_2 - d_1) \cdot (b^2 - b^1) \cdot \begin{bmatrix} a^1\gamma \\ 1-\gamma+a^1\gamma \end{bmatrix}, \tag{8}$$

$$\vec{p^*} - \vec{p^2} = \frac{\gamma}{1-\gamma} \cdot \frac{1}{1+(a^1-b^2)\gamma} \cdot \frac{1}{1+(a^2-b^2)\gamma} \cdot (d_2 - d_1) \cdot (a^1 - a^2) \cdot \begin{bmatrix} 1-b^2\gamma \\ (1-b^2)\gamma \end{bmatrix}, \tag{9}$$

$$\frac{\partial(\vec{p^*} - \vec{p^1})}{\partial b^1} < 0 < \frac{\partial(\vec{p^*} - \vec{p^1})}{\partial b^2}, \quad \frac{\partial(\vec{p^*} - \vec{p^2})}{\partial a^2} < 0 < \frac{\partial(\vec{p^*} - \vec{p^2})}{\partial a^1} \quad . \tag{10}$$

pf: Appendix A. *Q.E.D.*

This proposition demonstrates that there exist strictly positive speculative premiums in a Markovian framework unless there is an absolutely optimistic investor or the investors who hold the asset do not expected to sell it in the next period with probability 1, i.e., they do not expect to speculate at all. The later exception is described mathematically by the condition $a^1 \neq 0$ and $b^2 \neq 1$.

Before closing this section, we can apply our results to example 1. First we can construct a "representative belief" by equation (6) of Proposition 1,

$$Q^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}.$$

Then with Lemma 1 we know $(I - \gamma Q^*)$ is invertible and equation (5) in Proposition 1 gives us

$$\vec{p^*} = \begin{bmatrix} \frac{24}{13} \\ \frac{27}{13} \end{bmatrix} = \begin{bmatrix} 1.85 \\ 2.08 \end{bmatrix},$$

which is the same as the "minimal consistent price scheme" derived by Harrison and Kreps. In addition, we can analyze the properties of the risk premiums $\vec{p^*} - \vec{p^1}$ and $\vec{p^*} - \vec{p^2}$ by Proposition 2. Since there is no absolutely optimistic investor in this example, the speculative premiums are guaranteed by Proposition 2 to be positive. In the next section, we study how the results of this section can be generalized to the general case of $S \times S$ Markovian Beliefs.

3. Heterogeneous $S \times S$ Markovian Beliefs

In this section we will extend the results in the 2×2 Markov chains to the general case of $S \times S$ Markov chains. The first step is to show that $I - \gamma Q^i$ is still invertible as in Lemma 1.

Lemma 1′: For any $S \times S$ transition matrix Q^i, $(I - \gamma Q^i)$ is invertible.
pf: See Appendix A. *Q.E.D.*

With Lemma 1′, the subjective valuation by investors of type i as represented by equation (3) can also be solved for the general $S \times S$ case,

$$\vec{p^i} = (I - \gamma Q^i)^{-1} \gamma Q^i \vec{d}. \tag{4}$$

Following the same reasoning as in equation (7) and Proposition 1, we can find a type $i(s)$ of investors who have the highest valuation of the asset for any given state s. We will show that the market equilibrium price for the general $S \times S$ case can also be written as follows:

$$\vec{p^*} = \gamma \begin{bmatrix} q_1^{i(1)} \\ \cdot \\ q_s^{i(s)} \\ \cdot \\ q_S^{i(S)} \end{bmatrix} (\vec{p^*} + \vec{d}) = \gamma Q^* (\vec{p^*} + \vec{d}) \,, \tag{11}$$

where Q^* is the "representative belief" of a fictitious investor. This equilibrium price is exactly the same as the "minimal consistent price scheme" of Harrison and Kreps:

$$\vec{p^*} = \max_i \gamma Q^i (\vec{p^*} + \vec{d}) \ . \tag{7}$$

In contrast to the 2×2 case with a unique Q^* as in Proposition 1, the representative belief is not necessarily unique in the general case.

Example 2: In a 3×3 Markovian framework, there may exist multiple representative beliefs, but a unique equilibrium price still exists.

$$Q^1 = \begin{bmatrix} 0.5 & 0.4 & 0.1 \\ 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}, Q^2 = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.7 & 0.2 & 0.1 \end{bmatrix}, Q^3 = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.22 & 0.43 & 0.35 \end{bmatrix} .$$

Given the dividend vector $\vec{d} = [0, 0.5, 1]'$ and discount factor $\gamma = 0.75$, there is a unique equilibrium price $\vec{p^*}$ but two representative beliefs Q^{*1} and Q^{*2}.

$$\vec{p^*} = \begin{bmatrix} 1.6544 \\ 1.5221 \\ 1.6103 \end{bmatrix}, Q^{*1} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}, Q^{*2} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.22 & 0.43 & 0.35 \end{bmatrix} .$$

In fact, the third row of Q^{*1} is constructed from Q^1 and the third row of Q^{*2} is constructed from Q^3. When $s = 3$, both rows satisfy

$$p_3^* = \gamma q_3^i (\vec{p^*} + \vec{d}) = 1.6103 \,, \; i = 1, 3.$$

Now we can study the general case by considering the set of possible "combined beliefs" of some fictitious investor f, $f = 1, \cdots, K^S$:

$$\Psi = \{ Q^f | q_s^f = q_s^i \,, i = 1, \cdots K, s = 1, \cdots, S \} \ .$$

The equilibrium price $\vec{p^*}$ and representative belief Q^* can be discovered by the following algorithm:

Step 1: Compute the subjective valuation $\vec{p^i} = (I - \gamma Q^i)^{-1}\gamma Q^i \vec{d}$ if investors of type i are obliged to hold forever, $i = 1, \cdots, K$. For each s, find the highest valuation $p_s^{i0(s)}$ and its corresponding type $i0(s)$.

Construct $Q^{f0} = \begin{bmatrix} q_1^{i0(1)} \\ \vdots \\ q_S^{i0(S)} \end{bmatrix} \in \Psi.$

Step 2: Compute the corresponding price $\vec{p^{f0}} = (I - \gamma Q^{f0})^{-1}\gamma Q^{f0}\vec{d}$ for the fictitious belief Q^{f0} constructed in Step 1. Then compute the "willingness to pay" $\vec{W^{i0}} = \gamma Q^i(\vec{p^{f0}} + \vec{d})$ associated with $\vec{p^{f0}}$ for each type i. Find the highest willingness to pay for each state s and its corresponding type $i1(s)$. Set $Q^{f1} = \begin{bmatrix} q_1^{i1(1)} \\ \vdots \\ q_S^{i1(S)} \end{bmatrix} \in \Psi$, and Q^{f1} can be considered as a mapping from Q^{f0}. This defines a mapping F, $Q^{f1} = F(Q^{f0})$.

Step 3: Compute the corresponding price $\vec{p^{f1}} = (I - \gamma Q^{f1})^{-1}\gamma Q^{f1}\vec{d}$ for the fictitious belief Q^{f1} constructed in Step 2. If $\vec{p^{f1}} = \vec{p^{f0}}$, stop the algorithm and list the price $\vec{p^{f1}}$ and belief Q^{f1} as the equilibrium values. If $\vec{p^{f1}} \neq \vec{p^{f0}}$, repeat Step 2 until $\vec{p^{f(n+1)}} = \vec{p^{fn}} (= \vec{p^*})$. The corresponding $Q^{f(n+1)}$ and Q^{fn} are exactly the "representative beliefs" discussed before. Note that the representative beliefs are not unique while the equilibrium price is.

This algorithm searches through the elements of Ψ for a candidate for the representative belief. A fixed point is shown to exist in Proposition 3. Before stating our formal results, we can illustrate the finding of $\vec{p^*}$ and Q^* in the following examples:

Example 2(continued): Given the beliefs Q^i for $i = 1, 2, 3$, we first compute their subjective valuation if obliged to hold the asset forever:

$$\vec{p^1} = \begin{bmatrix} 0.9726 \\ 0.9726 \\ 1.2145 \end{bmatrix}, \vec{p^2} = \begin{bmatrix} 1.4069 \\ 1.3288 \\ 1.1762 \end{bmatrix}, \vec{p^3} = \begin{bmatrix} 1.2208 \\ 1.1690 \\ 1.3589 \end{bmatrix}.$$

We can find that $i0(1) = 2$, $i0(2) = 2$ and $i0(3) = 3$ are the types $i0(s)$ with highest valuation in state $s = 1, 2, 3$. Then we can construct Q^{f0} and compute $p^{\vec{f0}}$:

$$Q^{f0} = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.3 & 0.5 & 0.2 \\ 0.22 & 0.43 & 0.35 \end{bmatrix}, \quad p^{\vec{f0}} = \begin{bmatrix} 1.6544 \\ 1.5221 \\ 1.6103 \end{bmatrix} = \vec{p^*},$$

In fact, this algorithm converges in one step. In the next example, the algorithm converges in a finite number of steps.

Example 3: Assume that dividends and discount factor are the same as Example 2. Given

$$Q^1 = \begin{bmatrix} 0.41 & 0.44 & 0.15 \\ 0.48 & 0.09 & 0.43 \\ 0.60 & 0.23 & 0.17 \end{bmatrix}, Q^2 = \begin{bmatrix} 0.85 & 0.03 & 0.11 \\ 0.15 & 0.57 & 0.28 \\ 0.27 & 0.62 & 0.11 \end{bmatrix}, Q^3 = \begin{bmatrix} 0.21 & 0.29 & 0.50 \\ 0.04 & 0.82 & 0.14 \\ 0.64 & 0.25 & 0.12 \end{bmatrix},$$

we can get

$$\vec{p^1} = \begin{bmatrix} 1.14 \\ 1.20 \\ 1.07 \end{bmatrix}, \vec{p^2} = \begin{bmatrix} 0.63 \\ 1.27 \\ 1.12 \end{bmatrix}, \vec{p^3} = \begin{bmatrix} 1.57 \\ 1.56 \\ 1.33 \end{bmatrix}, Q^{f0} = \begin{bmatrix} 0.21 & 0.29 & 0.50 \\ 0.04 & 0.82 & 0.14 \\ 0.64 & 0.25 & 0.12 \end{bmatrix}, p^{\vec{f0}} = \vec{p^3}.$$

Next we find the willingness to pay given $p^{\vec{f0}}$, construct $Q^{f1} = F(Q^{f0})$ and compute $p^{\vec{f1}}$:

$$\vec{W^{10}} = \begin{bmatrix} 1.42 \\ 1.46 \\ 1.36 \end{bmatrix}, \vec{W^{20}} = \begin{bmatrix} 1.25 \\ 1.55 \\ 1.46 \end{bmatrix}, \vec{W^{30}} = \begin{bmatrix} 1.57 \\ 1.56 \\ 1.33 \end{bmatrix}, Q^{f1} = \begin{bmatrix} 0.21 & 0.29 & 0.50 \\ 0.04 & 0.82 & 0.14 \\ 0.27 & 0.62 & 0.11 \end{bmatrix}, p^{\vec{f1}} = \begin{bmatrix} 1.67 \\ 1.62 \\ 1.53 \end{bmatrix}.$$

Repeat this step, we have

$$\vec{W^{11}} = \begin{bmatrix} 1.50 \\ 1.56 \\ 1.44 \end{bmatrix}, \vec{W^{21}} = \begin{bmatrix} 1.34 \\ 1.63 \\ 1.53 \end{bmatrix}, \vec{W^{31}} = \begin{bmatrix} 1.67 \\ 1.62 \\ 1.41 \end{bmatrix}, Q^{f2} = \begin{bmatrix} 0.21 & 0.29 & 0.50 \\ 0.15 & 0.57 & 0.28 \\ 0.27 & 0.62 & 0.11 \end{bmatrix}, p^{\vec{f2}} = \begin{bmatrix} 1.68 \\ 1.63 \\ 1.54 \end{bmatrix}.$$

Then it can be shown that $Q^{f3} = Q^{f2}$ and hence $p^{\vec{f3}} = p^{\vec{f2}} = \vec{p^*}$.

In the next proposition, we summarize our findings and provide a rigorous proof for the existence of equilibrium price $\vec{p^*}$ and representative belief Q^*:

Proposition 3: For the $S \times S$ Markovian beliefs, there exists a unique market price $\vec{p^*}$ and at least one fictitious investor whose "representative belief" Q^* satisfies the following equation:

$$\vec{p^*} = (I - \gamma Q^*)^{-1}\gamma Q^* \vec{d}, \tag{12}$$

$$\text{and} \quad Q^* \in \Psi = \{Q^f | q_s^f = q_s^i, i = 1, \cdots K, s = 1, \cdots, S\}, \tag{13}$$

where Ψ is the set of possible "combined beliefs" of a fictitious investor.

pf: The subject valuation of a fictitious investor is

$$\vec{p^f} = (I - \gamma Q^f)^{-1}\gamma Q^f \vec{d} = \vec{p^f}(Q^f)\ , \ f = 1, \cdots, K^S\ . \tag{14}$$

Given $\vec{p^f}$, we can find the representative investor $i(s, \vec{p^f}(Q^f))$ who has the highest valuation at state s. This defines the following mapping from Ψ into itself:

$$F(Q^f) = \begin{bmatrix} q_1^{i(1,\vec{p^f}(Q^f))} \\ \cdot \\ q_s^{i(s,\vec{p^f}(Q^f))} \\ \cdot \\ q_S^{i(S,\vec{p^f}(Q^f))} \end{bmatrix}. \tag{15}$$

A fixed point of this mapping implies that the associated prices also have a fixed point $\vec{p^f} = \vec{p^*}$, which is the equilibrium or minimal consistent price scheme. There are only finite elements in Ψ. If there is no fixed point of F, then there must exist a cycle $Q^{f1} \cdots Q^{fM}$ such that $F(Q^{fm}) = F(Q^{f(m+1)})$ for $m \leq M - 1$ and $F(Q^{fM}) = Q^{f1}$. The corresponding prices are $\vec{p^{f1}} \cdots \vec{p^{fM}}$. We will show in the following that $\vec{p^{fm}} = \vec{p^*}$, $m = 1, \cdots, M$.

Since $F(Q^{f1}) = Q^{f2}$, as the price is equal to $\vec{p^{f1}} = \vec{p^f}(Q^f)$, the belief $q_s^{f2} = q_s^{i(s,\vec{p^{f1}}(Q^f))}$ gives the highest valuation $\gamma q_s^{f2}(\vec{p^{f1}} + \vec{d})$ of the asset and

$$\gamma q_s^{f2}(\vec{p^{f1}} + \vec{d}) \geq \gamma q_s^{f1}(\vec{p^{f1}} + \vec{d}) = p_s^{f1} \text{ for } s = 1, \cdots, S\,. \tag{16}$$

Hence we have

$$\begin{aligned} \gamma Q^{f2}(\vec{p^{f1}} + \vec{d}) &\geq \vec{p^{f1}}\,, \\ \text{or,} \ (I - \gamma Q^{f2})\vec{p^{f1}} &\leq \gamma Q^{f2}(\vec{d})\,. \end{aligned} \tag{17}$$

In addition,

$$(I - \gamma Q^{f2})\vec{p^{f2}} = \gamma Q^{f2}(\vec{d}) \,. \tag{18}$$

Combining (17) and (18), we have

$$(I - \gamma Q^{f2})(\vec{p^{f2}} - \vec{p^{f1}}) \geq 0 \,. \tag{19}$$

Suppose some component of $\vec{p^{f2}} - \vec{p^{f1}}$ is negative. Let $p_1^{f2} - p_1^{f1} < 0$ and $p_1^{f2} - p_1^{f1} \leq p_s^{f2} - p_s^{f1}$ for all s. Substituting it into (19), we obtain

$$\begin{aligned} & (1-\gamma q_{11}^{f2})(p_1^{f2} - p_1^{f1}) + (-\gamma q_{12}^{f2})(p_2^{f2} - p_2^{f1}) + \cdots + (-\gamma q_{1S}^{f2})(p_S^{f2} - p_S^{f1}) \\ = \;& (1-\gamma)(p_1^{f2} - p_1^{f1}) + \gamma(q_{12}^{f2}((p_1^{f2} - p_1^{f1}) - (p_2^{f2} - p_2^{f1})) + \cdots + q_{1S}^{f2}((p_1^{f2} - p_1^{f1}) - (p_S^{f2} - p_S^{f1}))) \\ <0 \;& , \end{aligned} \tag{20}$$

which contradicts (19). Hence all components of $\vec{p^{f2}} - \vec{p^{f1}}$ must be nonnegative. Therefore, $\vec{p^{f2}} \geq \vec{p^{f1}}$ and $\vec{p^{fm}}$ increases with m. Since $\vec{p^{fm}}$ forms a cycle, all these prices must be equal to a constant $\vec{p^{f}}$. Then fictitious beliefs $\{Q^{fm}\}$ all come with the same price $\vec{p^{f}} = \vec{p^{*}}$, which is the equilibrium or minimal consistent price scheme. We can also easily show the uniqueness of $\vec{p^{*}}$. $\qquad Q.E.D.$

Next we demonstrate that there are positive speculative premiums except in two extreme cases. One of the two exceptions is the presence of "absolutely optimistic" investors, which was defined in the last section. The other exception is the case when investors do not expect to sell it in the subsequent date with probability 1. With the general Markov chains, it is possible to get multiple "representative beliefs" in equilibrium, as illustrated above. Hence it is not straightforward to generalize the measure of the extent of heterogeneity of Proposition 2. However, we can use the difference between equilibrium price and investors' "willingness to pay" associated with $\vec{p^{*}}$ to represent the heterogeneity in the economy. Then we can show that the size of speculative premiums is an increasing function of the heterogeneity of the economy. Define

$$m_s^i = p_s^* - \gamma q_s^i(\vec{p^{*}} + \vec{d}) = \gamma(q_s^* - q_s^i)(\vec{p^{*}} + \vec{d}) \,, \tag{21}$$

which is dependent on the difference of beliefs $q_s^* - q_s^i$ and is evaluated with respect to $\vec{p^{*}} + \vec{d}$. This measures how the subjective valuation of the asset $\gamma q_s^i(\vec{p^{*}} + \vec{d})$ by investors of type i

deviates from the market valuation p_s^*. From equation (7) we know that $m_s^i \geq 0$. Define $\vec{m^i}$ to be $[m_1^i, \cdots, m_S^i]'$. Then we can demonstrate the relationship between speculative premiums and the extent of heterogeneity $\vec{m^i}$ in the following proposition.

Proposition 4: Suppose there is no absolutely optimistic investor in the market with the $S \times S$ Markovian beliefs. Then the speculative premium is positive ($p_s^* > p_s^i$ for all i) in each state s if there exists $q_{ss'}^{i(s)} \neq 0$ such that type $i(s)$ investors do not hold the asset when s' occurs. In addition, the size of speculative premium is an increasing function of the extent of heterogeneity measured by $\vec{m^i}$:

$$\vec{p^*} - \vec{p^i} = (I - \gamma Q^i)^{-1} \vec{m^i} \ . \tag{22}$$

pf: Suppose that state s occurs, the difference between the market price p_s^* and investors' evaluation is

$$\begin{aligned} p_s^* - p_s^i &= \gamma q_s^* (\vec{p^*} + \vec{d}) - \gamma q_s^i (\vec{p^i} + \vec{d}) \\ &= \gamma q_s^* (\vec{p^*} + \vec{d}) - \gamma q_s^i (\vec{p^*} + \vec{d}) + \gamma q_s^i (\vec{p^*} - \vec{p^i}) \\ &= m_s^i + \gamma q_s^i (\vec{p^*} - \vec{p^i}), \text{ for } \ s = 1, \cdots, S \, . \end{aligned}$$

Hence we have

$$\vec{p^*} - \vec{p^i} = \vec{m^i} + \gamma Q^i (\vec{p^*} - \vec{p^i})$$

By applying Lemma 1$'$, we can get

$$(I - \gamma Q^i)(\vec{p^*} - \vec{p^i}) = \vec{m^i} \geq 0 \, , \tag{23}$$

which implies that

$$\vec{p^*} - \vec{p^i} \geq 0 \, , \tag{24}$$

by an argument similar to the proof of proposition 3. At any state s we have the following inequality and equality:

$$\gamma q_s^i (\vec{p^*} + \vec{d}) \leq p_s^* \, ,$$

$$\gamma q_s^i (\vec{p^i} + \vec{d}) = p_s^i \, .$$

If type i investors do not hold the asset when state s occurs, the inequality is strict and

$$p_s^* - p_s^i > \gamma q_s^i (\vec{p^*} - \vec{p^i}) \, ,$$

Hence $p_s^* - p_s^i > 0$ by utilizing (24). If type i investors hold the asset, then $i(s) = i$ and

$$p_s^* - p_s^i = \gamma q_s^{i(s)}(\vec{p^*} - \vec{p^i}) \geq 0\,.$$

The above inequality is strict if $q_{ss'}^{i(s)} \neq 0$ for some s' and type i investors do not hold the asset when state s' occurs. So we have proved the first part of the proposition. As for the second part, we can derive equation (22) easily from equation (23). The positive correlation can be shown as follows. Since Q^i is assumed to be constant,

$$\triangle(\vec{p^*} - \vec{p^i}) = (I - \gamma Q^i)^{-1} \cdot \triangle(\vec{m^i})\,,$$

$$\text{or} \;\; (I - \gamma Q^i)\triangle(\vec{p^*} - \vec{p^i}) = \triangle(\vec{m^i}) \tag{25}$$

Suppose $\triangle(\vec{m^i}) \geq 0$. Then by an argument similar to the proof of Proposition 3, we can demonstrate that $\triangle(\vec{p^*} - \vec{p^i}) \geq 0$. Therefore, they are positively correlated. *Q.E.D.*

4. Endogenous Uncertainty with Rational Markovian Beliefs

In this section we will study how endogenous uncertainty may emerge with speculative trading. We will also provide arguments on why heterogeneous beliefs will persist in a Markovian framework with rational beliefs.

4.1 The Meaning of Endogenous Uncertainty

In the previous sections we found positive speculative premiums with sufficiently diverse beliefs. In the analysis we need not require the dividends to be distinct in all S states. The states $s = 1, \cdots, S$ can be used to represent possible values of market prices, of which some are affected by the exogenously given values of dividends. We say that "endogenous uncertainty" is present in a Markovian economy if the endogenous determined equilibrium prices are distinct even when the exogenous variables (dividends) take the same value (see Kurz and Wu(1996) and Huang and Wu(1999) for a formal definition and general discussion).

Utilizing the technique of constructing a "representative belief" as in the previous section, we first find conditions under which market equilibrium prices can be different even when dividends are the same. In the following example there are two possible values of dividends ($d_L = 0$, $d_H = 1$), but three distinct market equilibrium prices are present for $S = 3$. We define this phenomenon as the emergence of endogenous uncertainty.

Example 4: Suppose $S = 3$, $\gamma = 0.75$. The market equilibrium prices can be determined by the "representative beliefs" Q^* as in the last section. In equation (26) there are three distinct prices while there are only two prices in equation (27).

$$\vec{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, Q^* = \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.2 & 0.3 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}, \vec{p^*} = \begin{bmatrix} 1.995 \\ 2.171 \\ 2.089 \end{bmatrix}. \tag{26}$$

$$\vec{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, Q^* = \begin{bmatrix} 0.4 & 0.1 & 0.5 \\ 0.3 & 0.2 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}, \vec{p^*} = \begin{bmatrix} 1.946 \\ 2.027 \\ 2.027 \end{bmatrix}. \tag{27}$$

A closer examination reveals one possible reason for the differences between equation (26) and (27). In (26), the probability $Pr^*(d_{s^{t+1}} = 1|s^t = s)$ of getting high dividend ($d_H = 1$) given the current state s, according to the representative belief Q^*, is equal to $0.3 + 0.5 = 0.8$ when $s = 2$ and is equal to $0.3 + 0.4 = 0.7$ when $s = 3$. These two conditional probabilities are different. In (27), the two conditional probabilities given $s = 2$ and $s = 3$ are both equal to 0.7. Our observation can be formalized as a lemma, to be used later to characterize the conditions for the presence of endogenous uncertainty:

Lemma 2: Given S states and 2 possible values of dividends ($d = d_L, d_H$), the necessary and sufficient condition for the market equilibrium prices to be the same whenever the dividends are the same (i.e., the absence of endogenous uncertainty) is that the conditional probabilities of getting high (low) dividends are the same for all current states with high (low) dividends, according to the "representative beliefs" Q^*. This condition can be written as

$$Pr^*(d_{s^{t+1}} = d_H|s^t) = k_H \text{ , for all } s^t \text{ such that } d_{s^t} = d_H \text{ ,} \tag{28a}$$

$$\text{and} \quad Pr^*(d_{s^{t+1}} = d_L|s^t) = k_L \text{ , for all } s^t \text{ such that } d_{s^t} = d_L \text{ .} \tag{28b}$$

pf: Without loss of generality, let states $s = 1, \cdots \hat{s}$ correspond to the case of low dividend ($d_s = d_L$) and states $s = \hat{s}+1, \cdots S$, high dividend ($d_s = d_H$). The market equilibrium prices will satisfy

$$\begin{bmatrix} p_1 \\ \vdots \\ p_{\hat{s}} \\ p_{\hat{s}+1} \\ \vdots \\ p_S \end{bmatrix} = \gamma \begin{bmatrix} q^*_{1,1} & \cdots & q^*_{1,\hat{s}} & q^*_{1,\hat{s}+1} & \cdots & q^*_{1,S} \\ & & & \vdots & & \\ q^*_{\hat{s},1} & \cdots & q^*_{\hat{s},\hat{s}} & q^*_{\hat{s},\hat{s}+1} & \cdots & q^*_{\hat{s},S} \\ q^*_{\hat{s}+1,1} & \cdots & q^*_{\hat{s}+1,\hat{s}} & q^*_{\hat{s}+1,\hat{s}+1} & \cdots & q^*_{\hat{s}+1,S} \\ & & & \vdots & & \\ q^*_{S,1} & \cdots & q^*_{S,\hat{s}} & q^*_{S,\hat{s}+1} & \cdots & q^*_{S,S} \end{bmatrix} \left(\begin{bmatrix} p_1 \\ \vdots \\ p_{\hat{s}} \\ p_{\hat{s}+1} \\ \vdots \\ p_S \end{bmatrix} + \begin{bmatrix} d_L \\ \vdots \\ d_L \\ d_H \\ \vdots \\ d_H \end{bmatrix} \right) \tag{29}$$

The necessary part is proved first. Suppose $p^*_s = p^*_L$ for $s = 1, \cdots \hat{s}$ and $p^*_s = p^*_H$ for $s = \hat{s}+1, \cdots S$. Then equation (29) can be written as

$$\begin{bmatrix} p^*_L \\ \vdots \\ p^*_L \\ p^*_H \\ \vdots \\ p^*_H \end{bmatrix} = \gamma \begin{bmatrix} \sum_{j=1}^{\hat{s}} q^*_{1,j} & \sum_{j=\hat{s}+1}^{S} q^*_{1,j} \\ \vdots & \vdots \\ \sum_{j=1}^{\hat{s}} q^*_{\hat{s},j} & \sum_{j=\hat{s}+1}^{S} q^*_{\hat{s},j} \\ \sum_{j=1}^{\hat{s}} q^*_{\hat{s}+1,j} & \sum_{j=\hat{s}+1}^{S} q^*_{\hat{s}+1,j} \\ \vdots & \vdots \\ \sum_{j=1}^{\hat{s}} q^*_{S,j} & \sum_{j=\hat{s}+1}^{S} q^*_{S,j} \end{bmatrix} \begin{bmatrix} p^*_L + d_L \\ \vdots \\ p^*_L + d_L \\ p^*_H + d_H \\ \vdots \\ p^*_H + d_H \end{bmatrix},$$

which is equal to

$$\begin{bmatrix} p^*_L \\ p^*_H \end{bmatrix} = \gamma \begin{bmatrix} \sum_{j=1}^{\hat{s}} q^*_{s_1,j} & \sum_{j=\hat{s}+1}^{S} q^*_{s_1,j} \\ \sum_{j=1}^{\hat{s}} q^*_{s_2,j} & \sum_{j=\hat{s}+1}^{S} q^*_{s_2,j} \end{bmatrix} \begin{bmatrix} p^*_L + d_L \\ p^*_H + d_H \end{bmatrix}, \tag{30}$$

for any pair s_1 and s_2 where $s_1 = 1, \cdots \hat{s}$ and $s_2 = \hat{s}+1, \cdots S$. From our discussion on 2×2 Markovian case, the equilibrium price in (30) has a unique solution which is independent of s_1 and s_2. Hence,

$$\begin{aligned} & \textstyle\sum_{j=1}^{\hat{s}} q^*_{s,j} = k_L \text{, for all } s = 1, \cdots \hat{s}\,, \\ \text{and} \quad & \textstyle\sum_{j=\hat{s}+1}^{S} q^*_{s,j} = k_H \text{, for all } s = \hat{s}+1, \cdots S\,, \end{aligned} \tag{31}$$

which is equivalent to equation (28).

Next we prove the sufficiency part. Given that equation (28) is satisfied, then equation (29) can be reduced to

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \gamma \begin{bmatrix} k_L & 1-k_L \\ 1-k_H & k_H \end{bmatrix} \begin{bmatrix} p_L^* + d_L \\ p_H^* + d_H \end{bmatrix},$$

whose unique solution is also the solution to equation (29). The solution to equation (29) will be $p_s^* = p_L^*$ for $s_1 = 1, \cdots \hat{s}$ and $p_s^* = p_H^*$ for $s_2 = \hat{s}+1, \cdots S$. *Q.E.D.*

Lemma 2 applies to the case of two possible values of dividends($d_t \in D = \{d_L, d_H\}$), which is also assumed in this section. We also assume that the stationary measure of the dividend process follows a Markov chain which can be represented by a transition matrix of the form

$$\Lambda = \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix}. \tag{32}$$

If all investors possess structural knowledge as in equation (32), then from Proposition 1 the market prices should be represented by the matrix[1]

$$\vec{p^{\Lambda}} = \begin{bmatrix} p_L^{\Lambda} \\ p_H^{\Lambda} \end{bmatrix} = \frac{\gamma}{(1-\gamma)(1+\gamma-2\gamma\phi)} \cdot \begin{bmatrix} (\gamma - 2\gamma\phi + \phi)d_L + (1-\phi)d_H \\ (1-\phi)d_L + (\gamma - 2\gamma\phi + \phi)d_H \end{bmatrix}. \tag{33}$$

However, the investors generally do not possess structural knowledge, then they may form subjective beliefs about the process of the dividends and prices. This is in contrast to the rational expectations approach in economic modeling where all agents are assumed to be able to carry out necessary calculations to deduce the equilibrium price map $p_t = P(s_t)$ giving the knowledge of the exogenous state s_t. In fact, agents neither have structural knowledge nor have information about the beliefs of other agents. As the equilibrium price depends on the state of beliefs y_t of all agents, $p_t = P(s_t, y_t)$ fluctuates when the state of beliefs varies. The component of economic fluctuations which is due to the agents' beliefs hence represents an important kind of uncertainty faced by all agents. This is called "endogenous uncertainty"

[1] In general, structural knowledge includes precise information of demand or supply functions and probability laws, which are usually unobservable.

by Kurz and Wu(1996). In section 4.2 we will study the emergence of endogenous uncertainty where agents do not possess structural knowledge.

4.2 The Rational Belief Framework

We now introduce the framework of rational beliefs of Kurz(1994) and Kurz and Wu(1996) where agents do not have structural knowledge of the economy. If agents can form expectations without restrictions, we are in the framework of temporary equilibrium, which is criticized for its lack of rational utilization of information in learning. We allow rationality conditions to be imposed to put more restrictions on the system of heterogeneous Markovian beliefs we have analyzed so far. Agents are said to have "rational beliefs" if their beliefs cannot be refuted by data. In a non-stationary environment, agents can learn only about the stationary measure of the observed data. Agents agree only on the stationary measure of the environment, but they can still disagree on the timing of events or on the likelihood of some important and rare events even after exhausting all possibilities of learning. This provides a theoretic foundation for the continued presence of heterogeneous expectation even when learning is allowed. (A brief review of the theory of rational beliefs is contained in Appendix B.) In the rest of this section we study the properties of rational beliefs equilibria (RBE), which was also named as "RBE with social states of beliefs" by Kurz(1998). In contrast to the overlapping generation framework adopted by his paper, the agents are assumed to be infinitely lived in our model.

We assume that there are two types $i = 1, 2$ and many investors in each type[2]. The individual investor n of type i adopts a "state of belief" or an "assessment variable" $y^{i,n} \in Y^i = \{0, 1\}$, which is used to represent investor's private signal or state of mind. Investors of type i adopt matrix $Q^{i,0}$ as their belief when $y^{i,n} = 0$ and matrix $Q^{i,1}$ as their belief when $y^{i,n} = 1$. The probabilities of assessment variables for each agent are represented by $\text{Prob}\{y^{i,n} = 0\} = \alpha^i$ and $\text{Prob}\{y^{i,n} = 1\} = 1 - \alpha^i$. The vector $y = (y^{i,1}, \cdots, y^{1,N}, y^{2,1}, \cdots, y^{2,N})$ is a collection of individual states in an economy with $2N$ agents($i = 1, 2$, $n = 1, \cdots, N$). However, all agents of type i with the same assessment will have the same demand behavior. Following Cass, Chichilnisky and Wu(1996) and Kurz(1998), we can define a "social state" to include

[2]The results in this section hold for any finite number of types.

all those collections of individual states which yield the same aggregate demand for securities. The distribution of $y^{i,n}$ in a social state could be represented by $F^i = (f^i, 1 - f^i) \in \mathcal{F}^i$, $0 < f^i < 1$, where f^i is the proportion of type i investors who have assessment variables with the value $y^{i,n} = 1$ (the rest have assessment variables with the value $y^{i,n} = 0$). Let $\mathcal{F}^i$ be the space with J finite elements of F^i. For example, $\mathcal{F}^i = \{(0.8, 0.2), (0.2, 0.8)\}$ where each element is an F^i and $J = 2$. Then the state space is $D \times \mathcal{F}^1 \times \mathcal{F}^2$ with $S = 2 \cdot J^2$ elements. Each of the S elements is called a "social state". Note that the structure of the state space allows for possible correlation within types. For example, if the assessments are independent for all agents of type i with probability $\alpha^i = 0.5$, then $\mathcal{F}^i = \{(0.5, 0.5)\}$. On the other hand, if the assessments are perfectly correlated for all agents of type i, then there are two possible distributions and $\mathcal{F}^i = \{(0, 1),\ (1, 0)\}$. Other forms of correlation result in a non-degenerate distribution over the J possible elements of $\mathcal{F}^i$.

The stationary measure on $D \times \mathcal{F}^1 \times \mathcal{F}^2$ can be represented by the following transition matrix

$$\Gamma = \begin{bmatrix} \phi B & (1-\phi)B \\ (1-\phi)B & \phi B \end{bmatrix}, \tag{34}$$

where B is a $(J^2 \times J^2)$ transition matrix and the marginals of Γ on the dividend states D are represented by equation (32). We can represent the stationary transition matrix and belief matrices from S social states to S social states as

$$\Gamma = \begin{bmatrix} \Gamma_1 \\ \cdot \\ \Gamma_s \\ \cdot \\ \Gamma_S \end{bmatrix} = \begin{bmatrix} \Gamma_{11} & \cdots & \Gamma_{1S} \\ \vdots & \ddots & \vdots \\ \Gamma_{s1} & \cdots & \Gamma_{sS} \\ \vdots & \ddots & \vdots \\ \Gamma_{S1} & \cdots & \Gamma_{SS} \end{bmatrix}, \; Q^{i,j} = \begin{bmatrix} q_1^{i,j} \\ \cdot \\ q_s^{i,j} \\ \cdot \\ q_S^{i,j} \end{bmatrix} = \begin{bmatrix} q_{11}^{i,j} & \cdots & q_{1S}^{i,j} \\ \vdots & \ddots & \vdots \\ q_{s1}^{i,j} & \cdots & q_{sS}^{i,j} \\ \vdots & \ddots & \vdots \\ q_{S1}^{i,j} & \cdots & q_{SS}^{i,j} \end{bmatrix}, \quad \begin{array}{l} i = 1, 2, \\ j = 0, 1. \end{array} \tag{35}$$

The stationary measure Γ is computed from data, and the investors' beliefs will be made consistent with Γ. Then the rationality constraint of Kurz and Schneider(1996) in a Markovian framework requires that for each individual investor

$$\alpha^i Q^{i,0} + (1 - \alpha^i) Q^{i,1} = \Gamma \;\text{ for } i = 1, 2\,, \tag{36}$$

where $\alpha^i = \text{Prob}\{y^{i,n} = 0\}$ and $1 - \alpha^i = \text{Prob}\{y^{i,n} = 1\}$. The equilibrium achieved in such a framework with beliefs satisfying (36) is called a Rational Belief Equilibrium (RBE) with social states of beliefs.

Before proceeding with our analysis, we consider the benchmark case when all investors' beliefs coincide with the stationary measure, i.e. $Q^{i,j} = \Gamma$, $\forall i, j$. The market equilibrium price vector achieved in this benchmark case is denoted by $\vec{p^\Gamma}$. Then we can show that there exists no endogenous uncertainty and there are only two possible values of prices and $\vec{p^\Gamma}$ is reduced to $\vec{p^\Lambda}$ with structural knowledge represented by equation (33).

Lemma 3: Suppose $Q^{i,j} = \Gamma$ for all i, j. Then $\vec{p^\Gamma}$ has only two possible values p_L^Λ, p_H^Λ, associated with dividends $d_s = d_L$, d_H, respectively.

pf: The "representative belief" now is Γ. Since the sum of row vector of matrix B is equal to 1, Lemma 2 is satisfied with $k_H = \phi$ and $k_L = \phi$. Hence equation (30) can be written as

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \gamma \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \begin{bmatrix} p_L^* + d_L \\ p_H^* + d_H \end{bmatrix},$$

which has solution $p_H^* = p_H^\Lambda$, $p_L^* = p_L^\Lambda$ as in equation (33). *Q.E.D.*

In the framework of Harrison and Kreps(1978), the one-period belief Q^i is fixed for all periods under consideration. With a rational belief structure as specified by (36), investors may use different beliefs from time to time. Let the current belief at t be $Q^{i,j}$ by an investor of type i with $y^{i,n} = j$, then the beliefs in the future periods average out to be Γ, as required by (36):

$$\begin{array}{ccccc} \cdots & Q^{i,j} & \Gamma & \Gamma & \cdots \\ \hline | & | & | & | & | \\ & t & t+1 & t+2 & \end{array}$$

We can show that the expected revenue for sale in the next period is no less than the expected revenue for sale in later period. Therefore, in the rest of this section the highest valuation of any investor can be represented just by his/her expected revenue for sales in the next period, i.e., the left hand side of equation (37):

Lemma 4: For any investor of type i the following holds in an RBE:

$$\gamma Q^{i,j}(\vec{p^*} + \vec{d}) \geq \gamma Q^{i,j}(\gamma \Gamma(\vec{p^*} + \vec{d}) + \vec{d}) \tag{37}$$

pf: We prove it by contradiction. If the above inequality does not hold for some row, there exist some negative elements in $\gamma Q^{i,j}(\vec{p^*} - \gamma\Gamma(\vec{p^*} + \vec{d}))$. Because all elements of $Q^{i,j}$ are nonnegative, there must exist some s such that $p_s^* - \gamma\Gamma_s(\vec{p^*} + \vec{d}) < 0$. But in equilibrium

$$p_s^* \geq \gamma q_s^{i,j}(\vec{p^*} + \vec{d}) \text{ , for } s = 1, \cdots S \text{ and } i = 1, 2, \ j = 0, 1 \ . \tag{38}$$

If some of the above inequalities in equation (38) does not hold, then there exist some i, j, s such that

$$p_s^* < \gamma q_s^{i,j}(\vec{p^*} + \vec{d}) \ .$$

The investors of type i with $y^{i,n} = j$ can "speculate" by buying when state s occurs and selling it in the next period. From the inequalities in equation (38) and the rationality constraint in equation (36), we can obtain

$$p_s^* \geq \gamma\Gamma_s(\vec{p^*} + \vec{d}) \ .$$

This contradicts our hypothesis. So we have proved the lemma. *Q.E.D.*

From Lemma 4, in equilibrium the valuation of any investor can be represented just by his expected revenue from sales in the next period, as shown by following inequalities:

$$\begin{aligned} &\gamma Q^{i,j}(\vec{p^*} + \vec{d}) \\ &\geq \gamma Q^{i,j}(\gamma\Gamma(\vec{p^*} + \vec{d}) + \vec{d}) \\ &\geq \gamma Q^{i,j}(\gamma\Gamma(\gamma\Gamma(\vec{p^*} + \vec{d}) + \vec{d}) + \vec{d}) \\ &\cdots \end{aligned} \tag{39}$$

We can easily see that selling in later periods will not get any higher revenues than selling in the next period.

Lemma 4 serves an important role for the dynamic decision problem of speculation. As investors of type i adopt Rational Beliefs of matrices $Q^{i,0}, Q^{i,1}$, the probability measures of their subjective beliefs do not satisfy the assumption of Proposition 1 in Harrison and Kreps(1978)[3]. Lemma 4 allows us to consider only simple trading strategies of selling security in the next period instead of selling in later periods, even when the techniques of Harrison

[3]They proved this proposition by applying Doob's optional stopping theorem since investors' subject probability measures satisfy the property of smartingales (cf. Chung(1974)). However, in our framework the properties of smartingales are not satisfied when investors adopt Rational Beliefs.

and Kreps(1978) cannot be applied. So we can apply the technique and results developed in the previous sections even when the beliefs adopted by investors are not constant over time.

4.3 The Properties of Rational Belief Equilibrium

Now we can show that endogenous uncertainty generally exists in RBE, using the technique of constructing a "representative belief", as developed in the previous sections. We also demonstrate that the equilibrium prices can deviate from the prices p^{Γ} that would have been generated if agents had the structural knowledge represented by equation (33). The following examples illustrate the possibility of positive risk premiums once the assumption $Q^{i,j} = \Gamma$ for all i, j in Lemma 3 is dropped. Agents do not have structural knowledge, but they still try to learn from the data until the stationary measures of their beliefs coincide with Γ, i.e. their beliefs satisfy the rationality restrictions.

Example 5: Suppose that $D = \{1, 2\}$ and each $\mathcal{F}^i$ has two elements. The state space is $D \times \mathcal{F}^1 \times \mathcal{F}^2$ with eight elements. Let A be a 4×4 matrix with all elements equal to 0.25 and Γ be a 8×8 matrix with all elements equal to 0.125. To be consistent with equation (34), ϕ has to be 0.5. Let $\alpha^1 = \alpha^2 = 0.5$, $\gamma = 0.75$ and A_1 be a 1×4 matrix with all elements equal to 0.25. This is the case of i.i.d. type states with no correlation for the true distribution. Given

$$Q^{1,0} = \begin{bmatrix} 0.30A_1 & 0.70A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.60A_1 & 0.40A_1 \end{bmatrix}, Q^{1,1} = \begin{bmatrix} 0.70A_1 & 0.30A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.40A_1 & 0.60A_1 \end{bmatrix}, Q^{2,0} = \begin{bmatrix} 0.60A_1 & 0.40A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.45A_1 & 0.55A_1 \end{bmatrix}, Q^{2,1} = \begin{bmatrix} 0.40A_1 & 0.60A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.55A_1 & 0.45A_1 \end{bmatrix}.$$

We can check that these belief matrices satisfy the rationality restrictions (36) since $\alpha^1 = \frac{1}{2}$ and $\alpha^2 = \frac{1}{2}$: $\frac{1}{2}Q^{1,0} + \frac{1}{2}Q^{1,1} = \Gamma$, $\frac{1}{2}Q^{2,0} + \frac{1}{2}Q^{2,1} = \Gamma$. At any of the eight social states, all form of types of beliefs $\{Q^{1,1}, Q^{1,0}, Q^{2,1}, Q^{2,0}\}$ are present. Utilizing the techniques of constructing "representative beliefs" of Proposition 3, we can find $\vec{p^*}$ and Q^*:

$$Q^* = \begin{bmatrix} 0.3A_1 & 0.7A_1 \\ 0.2A_1 & 0.8A_1 \\ 0.4A_1 & 0.6A_1 \\ 0.4A_1 & 0.6A_1 \\ 0.3A_1 & 0.7A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.3A_1 & 0.7A_1 \\ 0.4A_1 & 0.6A_1 \end{bmatrix}, \quad \vec{p^*} = \begin{bmatrix} 5.0248 \\ 5.0991 \\ 4.9505 \\ 4.9505 \\ 5.0248 \\ 4.9876 \\ 5.0248 \\ 4.9505 \end{bmatrix} > \vec{p^\Gamma} = \begin{bmatrix} 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \end{bmatrix}.$$

In Example 5, the equilibrium prices exhibit the presence of "endogenous uncertainty" and positive risk premiums. However, there are cases without endogenous uncertainty, as illustrated in the following example.

Example 6: Given Γ, γ, A_1, A as specified in Example 5, but with different beliefs:

$$Q^{1,0} = \begin{bmatrix} 0.40A_1 & 0.60A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.30A_1 & 0.70A_1 \end{bmatrix}, Q^{1,1} = \begin{bmatrix} 0.60A_1 & 0.40A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.70A_1 & 0.30A_1 \end{bmatrix}, Q^{2,0} = \begin{bmatrix} 0.55A_1 & 0.45A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.60A_1 & 0.40A_1 \end{bmatrix}, Q^{2,1} = \begin{bmatrix} 0.45A_1 & 0.55A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.40A_1 & 0.60A_1 \end{bmatrix}.$$

We can check that these belief matrices satisfy the rationality restrictions with $\alpha_1 = \frac{1}{2}$ and

$\alpha_2 = \frac{1}{2}$. Using the same technique, we can find $\vec{p^*}$ and Q^*:

$$Q^* = \begin{bmatrix} 0.4A_1 & 0.6A_1 \\ 0.4A_1 & 0.6A_1 \\ 0.4A_1 & 0.6A_1 \\ 0.4A_1 & 0.6A_1 \\ 0.3A_1 & 0.7A_1 \\ 0.3A_1 & 0.7A_1 \\ 0.3A_1 & 0.7A_1 \\ 0.3A_1 & 0.7A_1 \end{bmatrix}, \qquad \vec{p^*} = \begin{bmatrix} 4.946 \\ 4.946 \\ 4.946 \\ 4.946 \\ 5.027 \\ 5.027 \\ 5.027 \\ 5.027 \end{bmatrix} > \vec{p^\Gamma}\ .$$

Note that in Example 6 there is no endogenous uncertainty, but the equilibrium price $\vec{p^*}$ is still not equal to $\vec{p^\Gamma}$ of the case with structural knowledge. The speculative premiums are still positive. This is called the "amplification effect" by Kurz(1998). Using the notation of Lemma 2, we find that $k_L = 0.4$ and $k_H = 0.7$, which are not equal to $\phi = 0.5$. In fact we can show that the case of no endogenous uncertainty degenerates to the case with structural knowledge if and only if $k_L = k_H = \phi$. In the following proposition, we also identify the conditions for the presence (or the absence) of endogenous uncertainty in a Rational Belief Equilibrium (RBE).

Proposition 5: There exists a unique RBE with social states of beliefs. In general there exists endogenous uncertainty. The necessary and sufficient conditions for nonexistence of endogenous uncertainty are

$$\sum_{k=1}^{\hat{s}} q^*_{sk} = k_L, \text{ for } s\text{=1 to } \hat{s}\,, \tag{40}$$

$$\sum_{k=\hat{s}+1}^{S} q^*_{sk} = k_H, \text{ for } s = \hat{s}+1 \text{ to } S\,, \tag{41}$$

where $s = 1, \cdots \hat{s}$ correspond to the case of low dividend, $s = \hat{s}+1, \cdots S$ correspond to the case of high dividend, $\hat{s} = \frac{S}{2}$ and Q^* with elements q^*_{sk} is the representative belief. When there is no endogenous uncertainty, the equilibrium price may not be equal to $\vec{p^\Gamma}$ of the case with structural knowledge. The case of no endogenous uncertainty degenerates to the case with structural knowledge if and only if $k_L = \phi$ and $k_H = \phi$.

pf: In general four types beliefs $\{Q^{1,1}, Q^{1,0}, Q^{2,1}, Q^{2,0}\}$ are all present. Proposition 3 holds for any given set of beliefs, so there exists (by Lemma 4) a unique equilibrium price $\vec{p^*}$ with a fictitious representative belief Q^*. As shown in Lemma 2, if (40) and (41) are not satisfied then there exists endogenous uncertainty. Only when (40) and (41) are satisfied then there is no endogenous uncertainty. In addition, we know

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \gamma \begin{bmatrix} k_L & 1-k_L \\ 1-k_H & k_H \end{bmatrix} \begin{bmatrix} p_L^* + d_L \\ p_H^* + d_H \end{bmatrix},$$

which may not have the same solution as $\vec{p^\Gamma}$, which solves the following equation

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \gamma \begin{bmatrix} \phi & 1-\phi \\ 1-\phi & \phi \end{bmatrix} \begin{bmatrix} p_L^* + d_L \\ p_H^* + d_H \end{bmatrix},$$

unless $k_L = \phi$, $k_H = \phi$. *Q.E.D.*

Note that the rationality restrictions do not affect the existence of equilibrium price $\vec{p^*}$ as we can see in Proposition 5, but will affect the level of $\vec{p^*}$. We will show later how the rationality restrictions give us positive premiums $\vec{p^*} - \vec{p^i} > 0$ as long as the beliefs of agents remain diverse enough. However, the equilibrium price may be smaller than p^Γ with structural knowledge if the agents' beliefs become perfectly correlated. When there are perfect correlations among the individual assessment variables $y^{i,n}$ within type i, that is, $y^{i,n} = y^i \in Y^i = \{0,1\}$, $\mathcal{F}^i = \{(0,1), (1,0)\}$ for all i, the state space can be represented by $D \times Y^1 \times Y^2$. We call the equilibrium of this case as "RBE with perfect correlation within types."

The true state space can be represented by the index mapping Φ:

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = \Phi \begin{bmatrix} d = d_L & y^1 = 0 & y^2 = 0 \\ d = d_L & y^1 = 0 & y^2 = 1 \\ d = d_L & y^1 = 1 & y^2 = 0 \\ d = d_L & y^1 = 1 & y^2 = 1 \\ d = d_H & y^1 = 0 & y^2 = 0 \\ d = d_H & y^1 = 0 & y^2 = 1 \\ d = d_H & y^1 = 1 & y^2 = 0 \\ d = d_H & y^1 = 1 & y^2 = 1 \end{bmatrix}. \tag{42}$$

The structure of the economy as represented by equation above is not known to investors in the economy, but the stationary measure Γ on $D \times Y^1 \times Y^2$ can be learned by agents:

$$\Gamma = \begin{bmatrix} \phi B & (1-\phi)B \\ (1-\phi)B & \phi B \end{bmatrix}, \tag{34}$$

where

$$B = [B_{ij}] = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} = \begin{bmatrix} a_1 & \alpha^1 - a_1 & \alpha^2 - a_1 & 1 + a_1 - \alpha^1 - \alpha^2 \\ a_2 & \alpha^1 - a_2 & \alpha^2 - a_2 & 1 + a_2 - \alpha^1 - \alpha^2 \\ a_3 & \alpha^1 - a_3 & \alpha^2 - a_3 & 1 + a_3 - \alpha^1 - \alpha^2 \\ a_4 & \alpha^1 - a_4 & \alpha^2 - a_4 & 1 + a_4 - \alpha^1 - \alpha^2 \end{bmatrix} \tag{43}$$

is a 4×4 transition matrix. Note that (43) implies that $\text{Prob}\{y^i = 0\} = \alpha^i$ for $i = 1, 2$, which is compatible with our specification for individual assessment variables. When state s occurs, type i investors adopt beliefs $Q^{i,0}$ in α^i proportion of days and $Q^{i,1}$ in other $1 - \alpha^i$ proportion of days. Notice that investors adopt two beliefs with each assessment variable y^i. However, with perfect correlation as described by equation (42), there is only one possible type of belief for each type of investors at any state. For example, at state $s = 1$, the investors of type 1 all have belief $Q^{1,0}$ ($y^1 = 0$) and investors of type 2 all have belief $Q^{2,0}$ ($y^2 = 0$). Next we show that equilibrium price may be greater or smaller than $p^{\vec{\Gamma}}$ of the case with structural knowledge for RBE with perfect correlation within types.

Example 7: Given Γ, γ, A_1 and A as specified in Example 5. Suppose

$$Q^{1,0} = \begin{bmatrix} 0.40A_1 & 0.60A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.30A_1 & 0.70A_1 \end{bmatrix}, Q^{1,1} = \begin{bmatrix} 0.60A_1 & 0.40A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.70A_1 & 0.30A_1 \end{bmatrix}, Q^{2,0} = \begin{bmatrix} 0.60A_1 & 0.40A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.60A_1 & 0.40A_1 \end{bmatrix}, Q^{2,1} = \begin{bmatrix} 0.40A_1 & 0.60A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.40A_1 & 0.60A_1 \end{bmatrix}.$$

Given perfect correlation as described by equation (42), at state $s = 1$ only the first row of $Q^{1,0}$ will become "effective" since state $s = 1$ is described by $y^1 = 0$. Hence we can construct the "effective beliefs" $Q^{i,e}$ of these two types of agents, $i = 1, 2$, and the corresponding "representative beliefs" based on the effective beliefs:

$$Q^{1,e} = \begin{bmatrix} 0.40A_1 & 0.60A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.70A_1 & 0.30A_1 \end{bmatrix}, Q^{2,e} = \begin{bmatrix} 0.60A_1 & 0.40A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.40A_1 & 0.60A_1 \end{bmatrix}, Q^{*} = \begin{bmatrix} 0.40A_1 & 0.60A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.40A_1 & 0.60A_1 \end{bmatrix}.$$

Applying the technique developed in the last section, we can find equilibrium price $\vec{p^*} > \vec{p^\Gamma}$.

$$\vec{p^*} = \begin{bmatrix} 4.8562 \\ 4.8937 \\ 4.8187 \\ 4.9312 \\ 4.9312 \\ 4.8187 \\ 4.8937 \\ 4.8563 \end{bmatrix} > \vec{p^\Gamma} = \begin{bmatrix} 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \end{bmatrix}.$$

We can also find cases under which the equilibrium prices p_s^* may be greater than p_s^Γ in some states, but smaller than p_s^Γ in other states. In the next example we show that the equilibrium prices can be smaller than $\vec{p^\Gamma}$ in all states.

Example 8: In a similar framework as the last example, we have the following beliefs:

$$Q^{1,0} = \begin{bmatrix} 0.55A_1 & 0.45A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.45A_1 & 0.55A_1 \\ 0.20A_1 & 0.80A_1 \end{bmatrix}, Q^{1,1} = \begin{bmatrix} 0.45A_1 & 0.55A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.35A_1 & 0.65A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.80A_1 & 0.20A_1 \end{bmatrix}, Q^{2,0} = \begin{bmatrix} 0.80A_1 & 0.20A_1 \\ 0.40A_1 & 0.60A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.40A_1 & 0.60A_1 \end{bmatrix}, Q^{2,1} = \begin{bmatrix} 0.20A_1 & 0.80A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.30A_1 & 0.70A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.20A_1 & 0.80A_1 \\ 0.60A_1 & 0.40A_1 \end{bmatrix}.$$

The effective beliefs $Q^{i,e}$ and the representative beliefs Q^* can be constructed:

$$Q^{1,e} = \begin{bmatrix} 0.55A_1 & 0.45A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.80A_1 & 0.20A_1 \end{bmatrix}, Q^{2,e} = \begin{bmatrix} 0.80A_1 & 0.20A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.80A_1 & 0.20A_1 \\ 0.60A_1 & 0.40A_1 \end{bmatrix}, Q^* = \begin{bmatrix} 0.55A_1 & 0.45A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.60A_1 & 0.40A_1 \\ 0.70A_1 & 0.30A_1 \\ 0.65A_1 & 0.35A_1 \\ 0.55A_1 & 0.45A_1 \\ 0.60A_1 & 0.40A_1 \end{bmatrix}.$$

Applying the technique developed before, we can find equilibrium price $\vec{p^*} < \vec{p^\Gamma}$.

$$\vec{p^*} = \begin{bmatrix} 4.1981 \\ 4.1610 \\ 4.0867 \\ 4.1610 \\ 4.0867 \\ 4.1238 \\ 4.1981 \\ 4.1610 \end{bmatrix} < \vec{p^\Gamma} = \begin{bmatrix} 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \\ 4.5 \end{bmatrix}.$$

Rationality Restrictions guarantee that agents sometimes are pessimistic and sometimes are optimistic. If all four matrices are present, we can get Q^* to be the most optimistic, then RBE will have $\vec{p^*} > \vec{p^\Gamma}$ and positive premiums exist. But for RBE with perfect correlation within types there are only two beliefs matrices are present at a time: $\{Q^{1,0}, Q^{2,0}\}$, $\{Q^{1,0}, Q^{2,1}\}$, $\{Q^{1,1}, Q^{2,0}\}$, $\{Q^{1,1}, Q^{2,1}\}$, Then this property may not hold.

The following proof holds in the general case when four beliefs matrices $\{Q^{1,0}, Q^{1,1}, Q^{2,0}, Q^{2,1}\}$ are all present. It also holds for the cases of $\{Q^{1,0}, Q^{1,1}, Q^{2,0}\}$ and $\{Q^{1,0}, Q^{2,0}, Q^{2,1}\}$, so long as one type has both belief matrices present in the economy. We call these social states: "RBE without perfect correlation within types".

Proposition 6: For RBE without perfect correlation within types, the equilibrium prices $\vec{p^*}$ are strictly greater than the prices $\vec{p^\Gamma}$ that would have been generated if agents had the structural knowledge if all elements of Γ are positive and $k_L \neq \phi$ or $k_H \neq \phi$. If $\vec{p^*}$ does not generate endogenous uncertainty(no more price states than dividend states), then $k_L \leq \phi$ and $k_H \geq \phi$ where k_L and k_H are derived in equations (40) and (41).

pf: For a given type i who has both belief matrices present in the economy,

$$\gamma Q^{i,j}(\vec{p^*} + \vec{d}) \leq \vec{p^*}, \text{ for } i = 0, 1\,. \tag{44}$$

The above inequality and the rationality constraints of equation (36) imply that

$$\gamma \Gamma(\vec{p^*} + \vec{d}) \leq \vec{p^*}\,.$$

We also know that $\gamma\Gamma(\vec{p^\Gamma} + \vec{d}) \leq \vec{p^\Gamma}$. Hence,

$$\vec{p^*} \geq (I - \gamma\Gamma)^{-1}\gamma\Gamma\vec{d} = \vec{p^\Gamma}\,.$$

In the non-degenerated case ($k_L \neq \phi$ or $k_H \neq \phi$) when all elements of Γ are nonzeros, we can show that $\vec{p^*}$ are strictly greater than $\vec{p^\Gamma}$. The rationality constraints also imply $k_L \leq \phi$ and $k_H \geq \phi$ when there is no endogenous uncertainty: $\vec{p^*} = [p_L^*, p_H^*]'$

$$\begin{bmatrix} p_L^* \\ p_H^* \end{bmatrix} = \left(I - \gamma \begin{bmatrix} k_L & 1-k_L \\ 1-k_H & k_H \end{bmatrix}\right)\gamma \begin{bmatrix} k_L & 1-k_L \\ 1-k_H & k_H \end{bmatrix}\vec{d} \tag{45}$$

$$\alpha_i \sum_{k=1}^{\hat{s}} q_{sk}^{i,0} + (1-\alpha_i) \sum_{k=1}^{\hat{s}} q_{sk}^{i,1} = \phi \ \text{ for } s = 1, \cdots, \hat{s}, \ \ i = 1, 2. \tag{46}$$

From (46), either $\sum_{k=1}^{\hat{s}} q_{sk}^{i,0}$ or $\sum_{k=1}^{\hat{s}} q_{sk}^{i,1}$ must be less or equal to ϕ. Let $\sum_{k=1}^{\hat{s}} q_{sk}^{i,0} \leq \phi$. Suppose $k_L > \phi$, then the willingness to pay by i is

$$\begin{array}{ll} \gamma q_s^{i,0}(\vec{p^*} + \vec{d}) & = \gamma \left(\sum_{k=1}^{\hat{s}} q_{sk}^{i,0}(p_L^* + d_L)(1 - \sum_{k=1}^{\hat{s}} q_{sk}^{i,0}(p_H^* + d_H)) \right) \\ & \gamma \left(k_L(p_L^* + d_L) + (1-k_L)(p_H^* + d_H) \right) = p_s^* \end{array}.$$

This leads the contradiction. If $\sum_{k=1}^{\hat{s}} q_{sk}^{i,1} \leq \phi$, we can also find $\gamma q_s^{i,0}(\vec{p^*} + \vec{d}) > p_s^*$. $Q.E.D.$

5. Concluding Remarks

As discussed by Morris(1996), using unmodeled heterogeneity of expectations in economics has been out of fashion. However, the continued presence of heterogeneous expectations observed in the real world remains an important phenomenon for us to understand. It is especially needed when we try to study the volatile behavior of speculative trading in financial markets. This paper proposes a research framework for modeling the presence of different beliefs which are rational and compatible with the data. Even with complete learning when all agents learn about the stationary measure, their beliefs can still remain diverse and non-stationary. Such a framework with heterogeneous but rational beliefs can help us to understand the functioning of speculative trading in the market place.

In particular, we find that Keynes' insight on speculation and subjective expectations can be supported in a rigorous framework. Our paper demonstrates that heterogeneous beliefs can be the major reason for speculative trading. The extent of expectations heterogeneity affects the size of speculative premiums. In such a framework, investors evaluate the asset according to its resale value and not just its dividend streams, just as described by Keynes. It hence provides a framework for pricing assets when speculative trading is allowed. It is also shown that endogenous uncertainty may emerge with speculative trading. The equilibrium prices are generally higher than the market fundamentals and speculative premiums are often positive. Although we adopt a simple framework, the basic results are still robust in a more general environment.

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Appendix A

Proof of Proposition 2:

pf:

$$\begin{aligned}
\vec{p^i} &= (I-\gamma Q^i)^{-1}\gamma Q^i \vec{d} \\
&= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \gamma \begin{bmatrix} 1-a^i & a^i \\ 1-b^i & b^i \end{bmatrix}\right]^{-1} \gamma \begin{bmatrix} 1-a^i & a^i \\ 1-b^i & b^i \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
&= \begin{bmatrix} 1-\gamma+\gamma a^i & -\gamma a^i \\ -\gamma+\gamma b^i & 1-\gamma b^i \end{bmatrix}^{-1} \gamma \begin{bmatrix} 1-a^i & a^i \\ 1-b^i & b^i \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
&= \tfrac{1}{(1-\gamma)(1+\gamma(a^i-b^i))} \cdot \begin{bmatrix} 1-\gamma b^i & \gamma a^i \\ \gamma-\gamma b^i & 1-\gamma+\gamma a^i \end{bmatrix} \gamma \begin{bmatrix} 1-a^i & a^i \\ 1-b^i & b^i \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
&= \tfrac{1}{(1-\gamma)(1+\gamma(a^i-b^i))} \begin{bmatrix} \gamma(1-a^i+a^i\gamma-b^i\gamma) & \gamma a^i \\ \gamma(1-b^i) & \gamma(b^i+a^i\gamma-b^i\gamma) \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \\
&= \tfrac{\gamma}{(1-\gamma)(1+\gamma(a^i-b^i))} \cdot \begin{bmatrix} (1-a^i+a^i\gamma-b^i\gamma)d_1 + a^i d_2 \\ (1-b^i)d_1 + (b^i+a^i\gamma-b^i\gamma)d_2 \end{bmatrix} .
\end{aligned} \tag{A.1}$$

And from Proposition 1,

$$Q^* = \begin{bmatrix} 1-\max_i a^i & \max_i a^i \\ 1-\max_i b^i & \max_i b^i \end{bmatrix} = \begin{bmatrix} 1-a^1 & a^1 \\ 1-b^2 & b^2 \end{bmatrix} , \tag{A.2}$$

we can obtain the formula of equilibrium prices

$$\vec{p^*} = \frac{\gamma}{(1-\gamma)(1+\gamma(a^1-b^2))} \cdot \begin{bmatrix} (1-a^1+a^1\gamma-b^2\gamma)d_1 + a^1 d_2 \\ (1-b^2)d_1 + (b^2+a^1\gamma-b^2\gamma)d_2 \end{bmatrix} . \tag{A.3}$$

By further calculation we can obtain equations (8) and (9) as in Proposition 2. From equations (8) and (9), we can obtain the four inequality as in (10).

$$\begin{aligned}
\tfrac{\partial(\vec{p^*}-\vec{p^1})}{\partial b^1} &= \tfrac{\gamma}{1-\gamma} \cdot \tfrac{1}{1+(a^1-b^2)\gamma} \cdot \tfrac{1}{(1+(a^1-b^1)\gamma)^2} \\
&\cdot (d_2-d_1) \cdot (-1-(a^1-b^2)\gamma) \cdot \begin{bmatrix} a^1\gamma \\ 1-\gamma+a^1\gamma \end{bmatrix} < 0 ,
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
\tfrac{\partial(\vec{p^*}-\vec{p^1})}{\partial b^2} &= \tfrac{\gamma}{1-\gamma} \cdot \tfrac{1}{(1+(a^1-b^2)\gamma)^\gamma} \cdot \tfrac{1}{1+(a^1-b^1)\gamma} \\
&\cdot (d_2-d_1) \cdot (1+(a^1-b^1)\gamma) \cdot \begin{bmatrix} a^1\gamma \\ 1-\gamma+a^1\gamma \end{bmatrix} > 0 ,
\end{aligned} \tag{A.5}$$

$$\frac{\partial(\vec{p^*}-\vec{p^2})}{\partial a^2} = \frac{\gamma}{1-\gamma}\cdot\frac{1}{1+(a^1-b^2)\gamma}\cdot\frac{1}{(1+(a^2-b^2)\gamma)^2} \cdot(d_2-d_1)\cdot(-1-(a^1-b^2)\gamma)\cdot\begin{bmatrix} 1-b^2\gamma \\ (1-b^2)\gamma \end{bmatrix} < 0\,, \tag{A.6}$$

$$\frac{\partial(\vec{p^*}-\vec{p^2})}{\partial a^1} = \frac{\gamma}{1-\gamma}\cdot\frac{1}{(1+(a^1-b^2)\gamma)^2}\cdot\frac{1}{1+(a^2-b^2)\gamma)} \cdot(d_2-d_1)\cdot(1+(a^2-b^1)\gamma)\cdot\begin{bmatrix} 1-b^2\gamma \\ (1-b^2)\gamma \end{bmatrix} > 0\,. \tag{A.7}$$

Q.E.D.

Proof of Lemma 1′:

pf: Let $B = I - \gamma Q^i$, the element $b_{ss'}$ of B is $1-\gamma q^i_{ss'}$ if $s = s'$ and $-\gamma q^i_{ss'}$ otherwise. If $\mathrm{Det}(I-\gamma Q^i) = \mathrm{Det}(B)$ were zero, then there exist $e_1, \cdots e_S$, not all zero, such that the weighting sum of the column vectors of the matrix B with the weight $\{e_s\}$ is not always equal to zero.

$$\sum_s e_s b_{s's} = 0 \qquad s' = 1, \cdots, S\,.$$

These are exactly the following equation,

$$\begin{array}{llllll} e_1(1-\gamma q^i_{11}) & +e_2(-\gamma q^i_{12}) & +\cdots & +e_S(-\gamma q^i_{1S}) & =0 \\ e_1(-\gamma q^i_{21}) & +e_2(1-\gamma q^i_{22}) & +\cdots & +e_S(-\gamma q^i_{2S}) & =0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_1(-\gamma q^i_{S1}) & +e_2(-\gamma q^i_{S2}) & +\cdots & +e_S(1-\gamma q^i_{SS}) & =0 \end{array}\,. \tag{A.8}$$

If there are positive numbers for $\{e_s\}$, we can choose a maximum (for S is finite) and assume it is e_1 which satisfies $e_1 \geq e_s$. Then from the above equation we can obtain the following,

$$\begin{aligned} 0 = {} & e_1(1-\gamma q^i_{11}) + e_2(-\gamma q^i_{12}) + \cdots + e_S(-\gamma q^i_{1S}) \\ = {} & e_1(1-\gamma) + e_1\gamma(1-q^i_{11}) + e_2(-\gamma q^i_{12}) + \cdots + e_S(-\gamma q^i_{1S}) \\ = {} & e_1(1-\gamma) + e_1\gamma(q^i_{12}+\cdots+q^i_{1S}) + e_2(-\gamma q^i_{12}) + \cdots + e_S(-\gamma q^i_{1S}) \\ = {} & e_1(1-\gamma) + \gamma(q^i_{12}(e_1-e_2) + q^i_{13}(e_1-e_3) + \cdots + q^i_{1S}(e_1-e_S)) \\ > {} & 0\,. \end{aligned} \tag{A.9}$$

So we have a contradiction. The proof is similar if there are negative numbers for $\{e_s\}$. In that case we choose the minimal number to apply a similar argument as above. So all elements of $\{e_s\}$ should be zero. This contradicts the hypothesis $\mathrm{Det}(I-\gamma Q^i) = 0$ and we have proved the lemma. *Q.E.D.*

Appendix B. A Brief Review of the Theory of Rational Beliefs

We start with some notation. $x_t \in \Re^N$ is a vector of N observables at date t and the sequence $\{x_t, t = 0, 1, \cdots\}$ is a stochastic process with true probability Π. Since every $x = (x_0, x_1, x_2, \cdots)$ is an infinite sequence in $(\Re^N)^\infty$ we use the notation $\Omega = (\Re^N)^\infty$ and denote by $\mathcal{F}$ the Borel σ-field of Ω. We thus think of the probability space $(\Omega, \mathcal{F}, \Pi)$ as the *true* probability space. A belief of an agent is a probability Q; such an agent is then adopting the theory that the probability space is $(\Omega, \mathcal{F}, Q)$. An agent who observes the data takes $(\Omega, \mathcal{F}, \Pi)$ as fixed but does not know Π. Using past data he will try to learn as much as possible about Π. The theory of Rational Beliefs aims to characterize the set of all beliefs which are compatible with the available data.

The basic assumption made is that date 1 has occurred a long time ago and at date t, when agents form their beliefs about the future beyond t, they have an *ample* supply of past data. We think of the vector $x = (x_0, x_1, x_2, x_3, \cdots)$ as the vector of observations generated by the economy. However, in studying complex joint distributions among the observables, econometricians consider *blocks* of data rather than individual, primitive observations. For example, if we study the distribution of $(x_{today}, x_{today+1})$ we would consider the infinite sequence of blocks $(x_0, x_1), (x_1, x_2), (x_2, x_3), \cdots$ It is thus useful to think of the data *from the perspective of data* 0 as the infinite vector $x = (x_0, x_1, x_2, \cdots)$ and the data *from the perspective of data t* as $x^t = (x_t, x_{t+1}, \cdots)$ where $x = x^0$ and

$$x^n = T\, x^{n-1} \, . \quad n = 1, 2, 3, \cdots$$

T is known as the *shift transformation*. In general the data from the perspective of data n is then $x^n = T^n x$. The stochastic dynamical system at hand is denoted by $(\Omega, \mathcal{F}, \Pi, T)$ where Π is the unknown probability. Now for any $B \in \mathcal{F}$ consider the set $T^{-n}B$ which is the preimage of B under T^n defined by

$$T^{-n}B = \{x \in \Omega \, : \, T^n x \in B\} \, .$$

$T^{-n}B$ is the set in Ω such that if we shift it by n dates we enter B; $T^{-n}B$ is the event B occurring n dates later. A system $(\Omega, \mathcal{F}, \Pi, T)$ is said to be *stationary* if $\Pi(B) = \Pi(T^{-1}B)$ for all $B \in \mathcal{F}$. A set $S \in \mathcal{F}$ is said to be *invariant* if $S = T^{-1}S$; it is said to be *invariant Π a.e.* if $\Pi(S \Delta T^{-1}S) = 0$,$(S \Delta T^{-1} S = (S \cup T^{-1}S) \backslash (S \cap T^{-1}S))$. The distinction between these two concepts of invariance are minimal and will be disregarded here. A dynamical system is said to be *ergodic* if $\Pi(S) = 1$ or $\Pi(S) = 0$ for any invariant set S. In the discussion below we assume for simplicity of exposition that $(\Omega, \mathcal{F}, \Pi, T)$ is ergodic but this assumption is not needed(see Kurz(1994) where this assumption is not made).

In order to learn probabilities agents adopt the natural way of studying the *frequencies* of all possible economic events. Using past data agents can compute for any finite dimensional set B the expression

$$m_n(B)(x) = \frac{1}{n}\sum_{k=0}^{n-1} 1_B(T^k x) = \left\{ \begin{array}{l} \text{The relative frequency that } B \text{ occurred among} \\ n \text{ observations since date } 0 \end{array} \right\}$$

where

$$1_B(y) = \begin{cases} 1 & \text{if } y \in B \\ 0 & \text{otherwise} \end{cases}$$

This leads to a definition of the basic property which the system $(\Omega, \mathcal{F}, \Pi, T)$ is assumed to have:

Definition B1: A dynamical system is called *stable* if for any finite dimensional set(i.e. cylinder) B

$$\lim_{n\to\infty} m_n(B)(x) = \overset{\circ}{m} \quad \text{exist} \quad \Pi \text{ a.e.} \tag{B.1}$$

The assumption of ergodicity ensures that the limit in Definition B1 is independent of x. In Kurz(1994) it is shown that the set function $\overset{\circ}{m}$ can be uniquely extended to a probability m on $(\Omega, \mathcal{F})$. Moreover, relative to this probability the dynamical system $(\Omega, \mathcal{F}, m, T)$ is stationary. There are two crucial observations for the theory of rational beliefs:

(a). Given the property of stability, in trying to learn Π all agents end up learning m which is a *stationary* probability. In general $m \neq \Pi$: the true dynamical system $(\Omega, \mathcal{F}, \Pi, T)$ may not be stationary. Π *cannot be learned.*

(b). Agents know that m may not be Π but with the data at hand m is the only thing that they can learn and agree upon.

Non-stationarity is a term which we employ to represent the process of *structural change which cannot be explained by the statistical regularity of past data.* Hence, a stable but non-stationary system is a model for an economy with structural change but in which econometric work can still be successfully carried out. If all agents knew that the true system is stationary they would adopt m as their belief. The problem is that they do not know if the environment is stationary and hence even if it was stationary, agents may still not adopt m as their belief.

It is important to see that m summarizes the entire collection of *asymptotic restrictions* imposed by the true system with probability Π on the empirical joint distributions of all the observed variables. It is shown in Kurz(1994) that for each stable system with probability Π there is an entire set $B(\Pi)$ of stable systems with probabilities Q which generate the same stationary probability m and consequently *impose the same asymptotic restrictions on the data as the true system with* Π. Kurz(1994) demonstrated that every stable system $(\Omega, \mathcal{F}, \Pi, T)$ generate a *unique* stationary probability m^{Π} which is *calculated analytically* from Π. This last fact is crucial since it is the foundation of the theory of rational beliefs:

Definition B2: A selection of belief Q cannot be contradicted by the data m if

(i). the system $(\Omega, \mathcal{F}, Q, T)$ is stable.

(ii). the system $(\Omega, \mathcal{F}, Q, T)$ generates m and hence $m^Q = m$.

We can finally state the two axioms which define the Rationality of beliefs:

Rationality Axioms: A selection of belief Q by an agent is a *Rational Belief* if it satisfies

Axiom I. *Compatibility with the Data:* Q cannot be contradicted by the data.

Axiom II. *Non-Degeneracy:* if $m(S) > 0$, then $Q(S) > 0$.

Rationality of belief requires that averaging the probabilities assigned to this event over all dates must yield the stationary probability assigned to it by m. However, non-stationary systems can give rise to an unbounded number of such events which are different from each other. Consequently, a Rational Belief Q may induce forecasts which are different from the forecasts of m at all dates and the difference between the forecasts of Q and m need not converge to zero.

Kurz(1994) demonstrated that two economic agents who are equally intelligent and who have *identically the same information* may make two different rational forecasts because they hold two competing theories which are compatible with the data. The agents may disagree on how much weight should be placed on the possibility that the environment is stationary. They may also disagree on the probabilities of time sequencing of events and on the likelihood of important and rare events. Disagreement among rational agents must, therefore, arise from their having different theories about *the nature of the fluctuations of the system* rather than about the behavior of its long term averages.