Back to basics: historical option pricing revisited

By Jean-Philippe Bouchaud[1,2] and Marc Potters[2]

[1]Service de Physique de l'État Condensé, Centre d'études de Saclay,
Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France
[2]Science & Finance, 109–111 rue Victor Hugo, 92523 Levallois Cedex, France

We reconsider the problem of option pricing using historical probability distributions. We first discuss how the risk-minimization scheme proposed recently is an adequate starting point under the realistic assumption that price increments are uncorrelated (but not necessarily independent) and of arbitrary probability density. We discuss in particular how, in the Gaussian limit, the Black–Scholes results are recovered, including the fact that the average return of the underlying stock disappears from the price (and the hedging strategy). We compare this theory to real option prices and find these reflect in a surprisingly accurate way the subtle statistical features of the underlying asset fluctuations.

Keywords: options; incomplete markets; volatility smile; implied kurtosis

1. Introduction

The famous Black–Scholes option pricing theory has two remarkable features: the hedging strategy eliminates risk entirely, and the option price does not depend at all on the average return of the underlying asset (Black & Scholes 1973; Hull 1997; Wilmott *et al*. 1993). The second property means that the option price is not simply the actualized average of the future pay-off over the historical probability distribution, which obviously would depend on the average return. This is even more striking in the case of the Cox–Ross–Rubinstein binomial model (Hull 1997; Cox *et al*. 1979) where the pricing measure is completely unrelated to the actual distribution of returns. This has lead to a rather abstract and general framework for derivative pricing, where the absence of arbitrage opportunities leads, for models where risk can be eliminated completely, to the existence of a 'risk-neutral probability measure' (unrelated to the historical one) over which the relevant average should be taken to obtain the price of derivatives (Harrison & Pliska 1981; Baxter & Rennie 1997). It is thus a rather common belief that the knowledge of the 'true' probability distribution of returns is a useless piece of information in order to price options. The credence is rather that the relevant 'implied' value of the parameters should be obtained from the option markets themselves, and used to price other instruments (for example, exotic options) (see, for example, Hull 1997; Dupire 1997).

However, in most models of stock fluctuations, except for very special cases (continuous-time Brownian motion and binomial, both being very poor representations of the reality), risk in option trading cannot be eliminated, and strict arbitrage opportunities do not exist, whatever the price of the option. That risk cannot be eliminated is furthermore the fundamental reason for the very existence of option markets. It

would thus be more satisfactory to have a theory of options where the true historical behaviour of the underlying asset was used to compute the option price, the hedging strategy, and *the residual risk*. The latter is clearly important to estimate, both for risk control purposes and also because it is reasonable to think that this residual risk partly determines the bid–ask spread imposed by market makers. The natural framework for this is the risk minimization approach developed by several authors (Schweizer 1994, 1999; Schäl 1994; Bouchaud & Sornette 1994; Bouchaud & Potters 1997; Aurell & Simdyankin 1998; Aurell & Życzkowski 1996; Wolczyńska 1998; Hammarlid 1998; Matacz 1998), where the optimal trading strategy is determined such that the chosen measure of risk (for example, the variance of the wealth balance) is minimized. The 'theoretical' price is then obtained using a fair-game argument. Note that in this approach the option price is not unique since it depends on the definition of risk; furthermore, a risk-premium correction to the fair-game price can be expected in general. From a theoretical point of view, this would generally be regarded as a lethal inconsistency. From a practical point of view, however, we see this as an advantage: since the price ambiguity is a constitutive property of option markets, it is interesting to understand the origin and size of this ambiguity. In this framework, the historical probability distribution determines the 'pricing kernel' to be used in the option price formula. We show in detail how, in the Black–Scholes limit, the average trend indeed completely disappears from the formula, and all the classical results are recovered. For more general models, however, the independence of the price on the average return is non-trivial.

The outline of this paper is as follows. We first recall the basic steps leading to option prices and optimal hedges for a general process with uncorrelated (but not necessarily independent) increments, which we present in terms of a cumulant expansion to show how the Black–Scholes results are obtained in the corresponding Gaussian limit. The first cumulant correction provides a theory for the volatility smile in terms of the (maturity dependent) kurtosis of the terminal price distribution. We compare this theory to real option prices (on a liquid market) and find that these option prices reflect in a surprisingly accurate way the subtle statistical features of the underlying asset fluctuations (Granger & Ding 1994; Guillaume *et al*. 1997; Cont *et al*. 1997; Bouchaud 1998), in particular the persistent nature of the volatility fluctuations.

2. A risk minimization theory of option pricing

(*a*) *The global wealth balance*

Let us first write the wealth balance equation corresponding to the writing of a European call option. At time $t = 0$, the writer receives the price of the option $\mathcal{C}[x_0, x_{\mathrm{s}}, T]$, on a certain asset whose value is $x(t = 0) = x_0$. The strike price is x_{s}. Between $t = 0$ and $t = T$, the writer trades the underlying asset at discrete times $t = k\tau$, $k = 1, \dots, N = T/\tau$; his strategy is to hold $\phi_k(x_k)$ assets if the price is $x(t) = x_k$ when the time is $t = k\tau$. It is easy to show that the change of wealth due to this trading is given by (Bouchaud & Potters 1997):

$$\Delta W_{\mathrm{trading}} = \sum_{k=0}^{N-1} \phi_k(x_k)[x_{k+1} - \mathrm{e}^{r\tau} x_k]\mathrm{e}^{r(T-t_k-\tau)}, \qquad (2.1)$$

where r is the risk-free rate and $t_k = k\tau$. At time $T = N\tau$, the writer loses the difference $x_N - x_s$ if the option is exercised. Thus the complete wealth balance reads

$$\Delta W = \mathcal{C}[x_0, x_s, T]\mathrm{e}^{rT} - \max(x_N - x_s, 0) + \sum_{k=0}^{N-1} \phi_k(x_k)\delta x_k \mathrm{e}^{r(T-t_{k+1})}, \qquad (2.2)$$

where we have introduced the notation $\delta x_k \equiv [x_{k+1} - \mathrm{e}^{r\tau}x_k]$. Note that δx_k is posterior to the instant k where ϕ_k is determined. Denoting as $\langle\cdots\rangle$ the average over the historical distribution, the average profit is given by

$$\langle\Delta W\rangle = \mathcal{C}[x_0, x_s, T]\mathrm{e}^{rT} - \langle\max(x_N - x_s, 0)\rangle + \sum_{k=0}^{N-1} \langle\phi_k(x_k)\rangle\langle\delta x_k\rangle\mathrm{e}^{r(T-t_{k+1})}. \quad (2.3)$$

The fair-game requirement then fixes $\mathcal{C}[x_0, x_s, T]$ such that $\langle\Delta W\rangle = 0$.

Although other interesting definitions could be considered (Bouchaud & Potters 1997), we restrict ourselves here to the case where the risk is measured as

$$R \equiv \langle\Delta W^2\rangle - \langle\Delta W\rangle^2 = \langle\Delta W^2\rangle. \qquad (2.4)$$

The risk R is always greater than or equal to zero and the minimum is obtained for a certain optimal strategy ϕ^*, determined by a functional derivation of (2.4) with respect to $\phi(x, t)$. This determines the option price through

$$\mathcal{C}[x_0, x_s, T] = \mathrm{e}^{-rT}\langle\max(x_N - x_s, 0)\rangle - \sum_{k=0}^{N-1} \langle\phi_k^*(x_k)\rangle\langle\delta x_k\rangle\mathrm{e}^{-rt_{k+1}}. \qquad (2.5)$$

Note that since ϕ^* depends *a priori* on our choice of the variance as the relevant measure of risk, the price of the option is not unique, but reflects (among other things) the operator's perception of risk.

(b) *The case of zero-excess average return*

In this section we consider the case where the average return of the stock over the bond, $m = \langle\delta x_k\rangle$, is zero. This simplifying hypothesis is often justified in practice for small maturities, where average return effects are small compared with volatilities, and can be treated perturbatively, as shown in §2c.

Let $P(x, T \mid x_0, 0)\,\mathrm{d}x$ be the probability that the asset value is x at time T, knowing that it was x_0 at time 0. When $m = 0$, equation (2.5) then yields an option price independent of the trading strategy:

$$\mathcal{C}[x_0, x_s, T; m = 0] = \mathrm{e}^{-rT}\langle\max(x_N - x_s, 0)\rangle \equiv \mathrm{e}^{-rT}\int_{x_s}^{\infty} \mathrm{d}x\,(x - x_s)P(x, T \mid x_0, 0).$$
$$(2.6)$$

Note that our assumption that $m = 0$ means that the average of $P(x, T \mid x_0, 0)$ is not at x_0, but at the forward price $x_0\mathrm{e}^{rT}$.

In order to proceed with the risk minimization, we shall assume that the price increments δx_k are uncorrelated random variables, such that $\langle\delta x_k\delta x_\ell\rangle = \sigma^2\delta_{k,\ell}$, where $\delta_{k,\ell}$ is the Kronecker symbol. The assumption that σ does not depend on k is in general not justified, since it amounts to assuming that share price follows an

additive random process of constant volatility (but with an arbitrary distribution for the increments). Actually, real data are often better described (for short maturities) as being an additive random process rather than a multiplicative one (Bouchaud & Potters 1997), an assumption which does introduce an spurious positive skew in the price distribution. In reality, however, σ depends on k, which reflects heteroskedastic effects (or time-persistent volatility (Granger & Ding 1994; Guillaume 1997)). Taking this effect into account would lead to more involved calculations, which can, however, still be completed analytically (Bouchaud & Potters 1997). With these approximations in mind, the relevant formula for risk is rather simple:

$$\langle \Delta W^2 \rangle = \langle \Delta W^2 \rangle_0 + \sigma^2 \sum_{k=0}^{N-1} \int_0^\infty \mathrm{d}x\, P(x, t_k \mid x_0, 0)\phi_k^2(x)\mathrm{e}^{2r(T-t_{k+1})}$$

$$- 2\mathrm{e}^{r(T-t_{k+1})} \sum_{k=0}^{N-1} \int_0^\infty \mathrm{d}x\, P(x, t_k \mid x_0, 0)\phi_k(x)$$

$$\times \int_{x_s}^\infty \mathrm{d}x'\, (x' - x_s)P(x', T \mid x, t_k)\langle \delta x_k \rangle_{x, t_k \to x', T}, \qquad (2.7)$$

where $\langle \Delta W^2 \rangle_0$ is the unhedged ($\phi_k \equiv 0$) risk associated with the option and

$$\langle \delta x_k \rangle_{x, t_k \to x', T}$$

is the conditional average of δx_k on the trajectories starting at x at time t_k and ending at point x' at time T.

The optimal trading strategy is obtained by setting (Bouchaud & Sornette 1994; Bouchaud & Potters 1997; Aurell & Simdyankin 1998; Aurell & Życzkowski 1996)

$$\frac{\partial \langle \Delta W^2 \rangle}{\partial \phi_k(x)} = 0 \qquad (2.8)$$

for all k and x. This leads to the following explicit result for the optimal hedging strategy:

$$\phi_k^*(x) = \frac{\mathrm{e}^{-r(T-t_{k+1})}}{\sigma^2} \int_{x_s}^\infty \mathrm{d}x'\, (x' - x_s)\langle \delta x_k \rangle_{x, t_k \to x', T} P(x', T \mid x, t_k). \qquad (2.9)$$

This formula simplifies somewhat when the increments are Gaussian, and one finally finds the famous Black–Scholes 'Δ-hedge':

$$\phi_k^*(x) = \partial \mathcal{C}[x, x_s, T - t_k]/\partial x.$$

In the non-Gaussian case, however, this simple relation between the derivative of the option price and the trading strategy no longer holds (see equation (2.13) below).

Inserting (2.9) into (2.7) leads to the following formula for the residual risk:

$$R^* = \langle \Delta W^2 \rangle_0 - D\tau \sum_{k=0}^{N-1} \int_0^\infty \mathrm{d}x\, P(x, t_k \mid x_0, 0)\phi_k^{*2}(x)\mathrm{e}^{-r(T-t_{k+1})}. \qquad (2.10)$$

In general, the left-hand side of (2.10) is non-zero; in practice it is quite high— for example, for typical one-month options on liquid markets, $\sqrt{R*}$ represents as

much as 25% of the option price itself (Bouchaud & Potters 1997). However, in the special case where $P(x, t \mid x_0, 0)$ is normal (or lognormal), and in the limit of continuous trading, that is, when $\tau \to 0$, one can show that the residual risk R^* actually vanishes, thanks to a somewhat miraculous identity for Gaussian integrals (Bouchaud & Sornette 1994). Hence, the above formalism matches smoothly with all the Black–Scholes results in the limit of a continuous-time Brownian (or log-Brownian) process, at least when the excess average return of the asset is zero. Let us now discuss how these results are changed if the average return $m \equiv \langle \delta x_k \rangle$ is non-zero (but small).

(c) *Small non-zero average return*

More precisely, we shall consider the case where $mN \ll \sigma\sqrt{N}$ ($N = T/\tau$), or, more intuitively, that the average return on the time-scale of the option is small compared with the typical variations, which is certainly the case for options up to a few months.† The global wealth balance then includes the term related to the trading strategy, which reads‡

$$\langle \Delta W_{\text{trading}} \rangle = m \sum_{k=0}^{N-1} \int_0^\infty \mathrm{d}x \, P(x, t = k\tau \mid x_0, 0) \phi_k^*(x). \qquad (2.11)$$

The advantage of considering a small average return is that one can do a perturbation around the zero-average return case, and still use the explicit optimal strategy of (2.9) to lowest order in m.

Compared to the case $m = 0$, the option price is changed both because $P(x, k \mid x_0, 0)$ is *biased*, and because $\langle \Delta W_{\text{trading}} \rangle$ must be substracted from equation (2.5). It is convenient to use the Fourier transform of the probability distribution

$$\tilde{P}(z) = \int_{-\infty}^\infty \mathrm{d}x \, P(x, N \mid x_0, 0) \exp(\mathrm{i}xz)$$

and to expand it in a series introducing the *cumulants* c_n. They are defined by

$$\tilde{P}(z) = \exp\left[\sum_{n=1}^\infty \frac{c_n(\mathrm{i}z)^n}{n!} \right], \qquad (2.12)$$

where $c_1 = mN$, $c_2 = N\sigma^2$, $\kappa = c_4/c_2^2$ is the kurtosis, etc. Applying the cumulant expansion to the probability distribution in equation (2.9), we obtain the following optimal strategy for $m = 0$ (Bouchaud & Potters 1997):

$$\phi_k^*(x) = \frac{1}{c_2} \sum_{n=2}^\infty \frac{(-1)^n c_n}{(n-1)!} \frac{\partial^{n-1}}{\partial x^{n-1}} \mathcal{C}[x, x_{\text{s}}, T - t_k]. \qquad (2.13)$$

Note again that in the Gaussian case ($c_n = 0$ for $n > 2$), one recovers the standard Black–Scholes 'Δ-hedge'.

† Typically, $m = 5\%$ annual and $\sigma = 15\%$ annual. The order of magnitude of the error made in neglecting the second-order term in m is $m^2 N/\sigma^2 \simeq 0.1$ even for $N\tau = 1$ year.

‡ For the sake of simplicity, we shall set the interest r to zero in the following. See Aurell & Simdyankin (1998) and Aurell & Życzkowski (1996) for a more complete discussion.

Inserting the optimal strategy (2.13) into equation (2.11) and integrating by parts one gets an expansion of the trading term $\langle \Delta W_{\text{trading}} \rangle$. Inserted into equation (2.5), this gives the following correction to the option price (Bouchaud & Potters 1997):

$$\mathcal{C}[x_0, x_s, T; m] = \mathcal{C}[x_0, x_s, T; m = 0]$$

$$- \frac{m}{c_2} \sum_{n=3}^{\infty} \frac{c_n}{(n-1)!} \frac{\partial^{n-3}}{\partial x_0^{n-3}} P_0(x', N | x_0, 0)|_{x'=x_s} + \mathcal{O}(m^2), \quad (2.14)$$

where the shorthand P_0 stands for the probability distribution where the first cumulant m has been set to zero.

In the Gaussian case, $c_n = 0$ for all $n \geqslant 3$, and one thus sees explicitly that $\mathcal{C}_m = \mathcal{C}_0$, at least to first order in m. Actually, one can show that this is true to all orders in m in the Gaussian case, which is an alternative way to derive the result of Black & Scholes in a Gaussian context (Bouchaud & Potters 1997).†

However, for even distributions with fat tails ($c_3 = 0$ and $c_4 > 0$), it is easy to see from equation (2.14) that a positive average return $m > 0$ decreases the price of out-of-the-money options ($x_s > x_0$), and increases the price of in-the-money options ($x_s < x_0$). Hence, we again see explicitly that the independence of the option price on the average return m, which is one of the most important results of Black & Scholes, does not survive for more general models of stock fluctuations.

Note finally that equation (2.14) can also be written as

$$\mathcal{C}[x_0, x_s, T; m] = \int_{x_s}^{\infty} \mathrm{d}x \, (x - x_s) Q(x, T \mid x_0, 0) \quad (2.15)$$

with an effective distribution Q defined as

$$Q(x, T \mid x_0, 0) = P_0(x, T \mid x_0, 0) - \frac{m}{c_2} \sum_{n=3}^{\infty} \frac{c_n}{(n-1)!} \frac{\partial^{n-1}}{\partial x_0^{n-1}} P_0(x, N \mid x_0, 0). \quad (2.16)$$

The integral over x of Q is one, but Q is not *a priori* positive everywhere. This means that for a certain family of payoffs, the fair price of the option (2.15) may be negative in the absence of risk premium. From a practical point of view, however, this requires rather absurd values for the average return and for the strike price, which in turn would lead to a large residual risk.

The pseudo-distribution Q generalizes the 'risk neutral probability' usually discussed in the context of the Black–Scholes theory, and also has the property that the excess average return (the integral of $(x - x_0)Q$ over x) is zero, as can easily be seen by inspection from (2.16). In fact, one can derive a general formula for Q without any restriction on m of r (Aurell & Simdyankin 1998; Aurell & Życzkowski 1996; Wolczyńska 1998; Hammarlid 1998), and the effective distribution still has the properties that it is normalized to one and has zero-average excess return.

3. Volatility smile and implied kurtosis

In the case where the market fluctuations are moderately non-Gaussian, as is the case for liquid markets, one might expect that the first terms in the cumulant expansion

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† This result is obvious in the framework of Ito's calculus, which is only valid for all Gaussian processes (including the lognormal) in the continuum time limit.
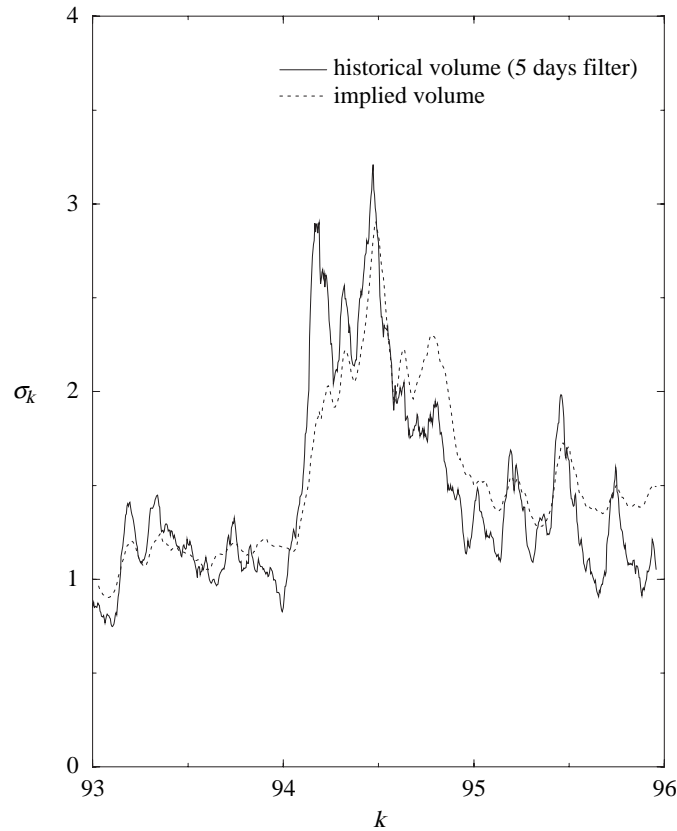
Figure 1. Comparison between the historical volatility of the BUND (measured from high-frequency data and filtered over the past five days), and the implied volatility, extracted from the option prices through formula (3.1).

around the Black–Scholes formula are sufficient to account for real option prices. If one only retains the leading-order correction which is (for symmetric fluctuations) proportional to the kurtosis κ, one finds that the price of options $\mathcal{C}(x_0, x_\mathrm{s}, T)$ can be written as a Gaussian Black–Scholes formula,† but with a modified value of the volatility σ, which becomes price and maturity dependent (Potters *et al.* 1998):

$$\sigma_\mathrm{imp}(x_\mathrm{s}, T) = \sigma\left[1 + \tfrac{1}{24}\kappa_T\left(\frac{(x_\mathrm{s} - x_0)^2}{\sigma^2 T} - 1\right)\right]. \tag{3.1}$$

The volatility σ_imp is called the implied volatility by the market operators, who use the standard Black–Scholes formula to price options, but with a value of the volatility which they estimate intuitively, and which turns out to depend on the exercise price in a roughly parabolic manner, as indeed suggested by equation (3.1). This is the famous 'volatility smile'. Equation (3.1) furthermore shows that the curvature of the smile is directly related to the kurtosis κ_T of the underlying statistical process on the

† Note that the operators use the rather more standard lognormal Black–Scholes formula, which, as noted above, induces a spurious positive skew not present in real data (at least for short maturities). In order to correct for this skew, the lognormal volatility smile is then negatively skewed. A more symmetric smile is observed if one talks in terms of a Gaussian volatility, which is what we adopt in the following.
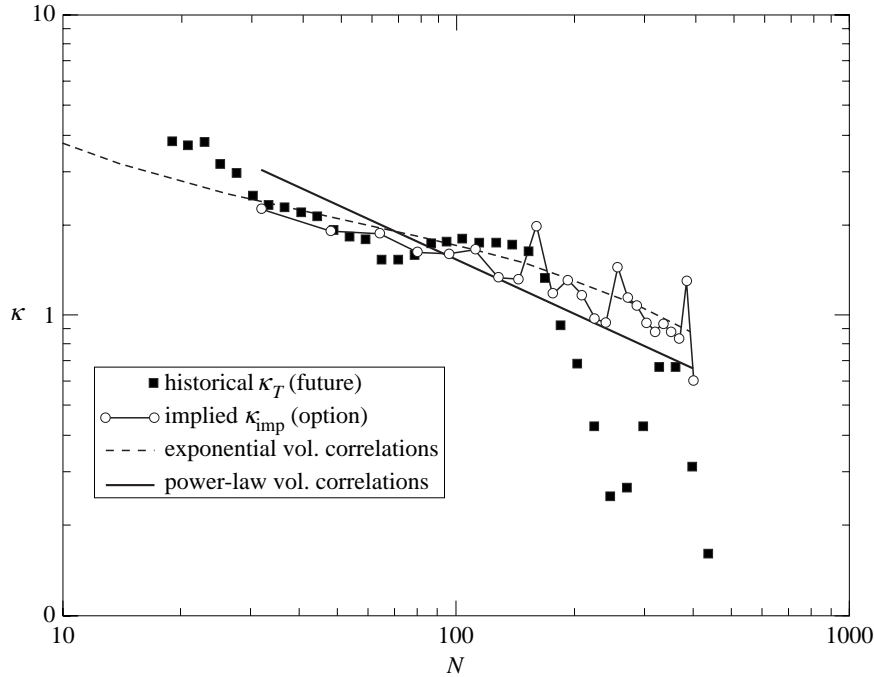
Figure 2. Plot (in log–log coordinates) of the average implied kurtosis κ_{imp} (determined by fitting the implied volatility for a fixed maturity by a parabola) and of the empirical kurtosis κ_N (determined directly from the historical movements of the BUND contract), as a function of the reduced time-scale $N = T/\tau$, $\tau = 30$ min. All transactions of options on the BUND future from 1993 to 1995 were analysed along with 5 min tick data of the BUND future for the same period. We show for comparison a fit with $\kappa_N \simeq N^{-0.6}$ (dark line). A fit with an exponentially decaying volatility correlation function is, however, also acceptable (dotted line).

scale of the maturity $T = N\tau$. We have tested this prediction by directly comparing the 'implied kurtosis',† obtained by extracting from real option prices the volatility σ and the curvature of the implied volatility smile, with the historical value of the volatility and of the kurtosis κ_N. We have mostly studied short-maturities (up to two months) options on futures, for which the interest rate can be set to zero. We also restrict ourselves to liquid markets (such as the BUND option market) where: (i) non-Gaussian effects are not too strong; and (ii) risk premiums are expected to be small, and thus where a comparison with the fair price is meaningful.

We have found the following results. The implied volatility turns out to be highly correlated with a short time filter of the historical volatility (see figure 1). Figure 2 shows the comparison between the implied kurtosis and historical kurtosis, *with no further ajustable parameters*. Note that the historical kurtosis decays more slowly than N^{-1}, which would be expected for a process with independent identically distributed increments. This anomalously slow decay is directly related to volatility persistence effects (Potters *et al.* 1998; Bouchaud & Potters 1997).

† The idea of an implied kurtosis is also discussed in Jarrow & Rudd (1982) and Corrado & Su (1996). These authors, however, considered an expansion around the lognormal distribution rather than around a Gaussian distribution, as discussed here (see also the previous footnote).

It is interesting to note that the kurtosis correction to the optimal strategy does not coincide with the market practice of using the implied volatility in the Black–Scholes Δ-hedge. However, since the risk is minimum for $\phi = \phi^*$, this means that the increase of risk due to a small error $\delta\phi$ in the strategy is only of order $\delta\phi^2$, and thus often quite small in practical applications.

The remarkable agreement between the implied and historical value of the parameters (which we have also found on a variety of other assets), and the fact that they evolve similarly with maturity, shows that the market as a whole is able to correct (by trial and error) the inadequacies of the Black–Scholes formula, and to encode in a satisfactory way both the fact that the distribution has a positive kurtosis, and that this kurtosis decays with maturity in an anomalous fashion due to volatility persistence effects.

4. Conclusion

In our opinion, mathematical finance in the past decades has overfocused on the concept of arbitrage-free pricing, which relies on very specific models (or instruments) where risk can be eliminated completely. This leads to a remarkably elegant and consistent formalism, where derivative pricing amounts to determining the risk-neutral probability measure, which in general does not coincide with the historical measure. In doing so, however, many important and subtle features are swept under the rug, in particular the amplitude of the residual risk. Furthermore, the fact that the risk-neutral and historical probabilities need not be the same is often an excuse for not worrying when the parameters of a specific model deduced from derivative markets are very different from historical parameters. This is particularly obvious in the case of interest rates (Bouchaud *et al.* 1997, 1998). In our mind, this rather reflects that an important effect has been left out of the models, which in the case of interest rates is a risk premium effect (Bouchaud *et al.* 1997, 1998). We believe that a more versatile (although less elegant from a mathematical point of view) theory of derivative pricing, such as the one discussed above, allows one to use, in a consistent and fruitful way, the empirical data on the underlying asset to price, hedge and control the risk of the corresponding derivative security. Extension of these ideas to interest rate derivatives is underway.

References

Aurell, E. & Simdyankin, S. 1998 *Int. Jl Theor. Appl. Finance* **1**, 1.

Aurell, E. & Życzkowski, K. 1996 Preprint, ewp-fin/9601001 at http://econwpa.wustl.edu/wpawelcome.html

Baxter, M. & Rennie, A. 1997 *Financial calculus.* Cambridge University Press.

Black, F. & Scholes, M. 1973 *J. Political Econ.* **81**, 637.

Bouchaud, J. P. 1998 Elements for a theory of financial risks. Preprint, http://xxx.lanl.gov/cond-mat/9806101.

Bouchaud, J. P. & Potters, M. 1997 *Theory of financial risk.* Paris: Aléa-Saclay, Eyrolles (in french). English translation available at http://www.science-finance.fr.

Bouchaud, J. P. & Sornette, D. 1994 *J. Physique* **4**, 863.

Bouchaud, J. P., Sagna, N., Cont, R., ElKaroui, N. & Potters, M. 1997 Phenomenology of the interest rate curve. Preprint, http://xxx.lanl.gov/cond-mat/9712164. *Appl. Math. Finance.* (In the press.)

Bouchaud, J. P., Sagna, N., Cont, R., ElKaroui, N. & Potters, M. 1998 Strings attached. *Risk Magazine* July, 56–59.

Cont, R., Potters, M. & Bouchaud, J. P. 1997 Scaling in stock market data: stable laws and beyond. In *Scale invariance and beyond* (ed. B. Dubrulle, F. Graner & D. Sornette). EDP Sciences.

Corrado, C. J. & Su, T. 1996 *J. Financial Res.* **19**, 175.

Cox, J., Ross, S. & Rubinstein, M. 1979 *J. Financial Econ.* **7**, 229.

Dupire, B. 1997 In *Mathematics of derivative securities* (ed. M. Dempster & S. Pliska). Cambridge University Press.

Granger, C. W. J. & Ding, Z. X. 1994 Stylized facts on the temporal distributional properties of daily data from speculative markets. Working paper, University of California, San Diego.

Guillaume, D. M., Dacorogna, M. M., Davé, R. R., Müller, U. A., Olsen, R. B. & Pictet, O. V. 1997 From the bird's eye to the microscope: a survey of new stylized facts of the intra-day foreign exchange markets. *Finance Stochastics* **1**, 95–130.

Hammarlid, O. 1998 *Int. Jl Theor. Appl. Finance* **1** 227.

Harrison, M. & Pliska, S. 1981 *Stochastic Processes Applic.* **11**, 215.

Hull, J. C. 1997 *Futures, options and other derivative securities.* Englewood Cliffs, NJ: Prentice Hall.

Jarrow, R. & Rudd, A. 1982 *J. Financial Econ.* **10**, 347.

Matacz, A. 1998 Preprint, http://xxx.lanl.gov/cond-mat/9710197. *Int. Jl Theor. Appl. Finance.* (In the press.)

Potters, M., Cont, R. & Bouchaud, J. P. 1998 *Europhys. Lett.* **41**, 239.

Schäl, M. 1994 *Math. Operations Res.* **19**, 121.

Schweizer, M. 1994 *Annals Prob.* **22**, 1536.

Schweizer, M. 1999 *Int. Jl Theor. Appl. Finance* **2**, 59.

Wilmott, P., Dewynne, J. N. & Howison, S. D. 1993 *Option pricing, mathematical methods and computation.* Oxford University Press.

Wolczyńska, G. 1998 *Int. Jl Theor. Appl. Finance* **1**, 283.