Credit Risk Modelling and Credit Derivatives

Inaugural-Dissertation

zur Erlangung des Grades eines Doktors

der Wirtschafts- und Gesellschaftswissenschaften

durch die

Rechts- und Staatswissenschaftliche Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Philipp J. Schönbucher

aus **Düsseldorf**

BONN 2000

Dekan:	Prof. Dr. Rüdiger Breuer
Erstreferent:	Prof. Dr. Dieter Sondermann
Zweitreferent:	Prof. Dr. Klaus Schürger

Tag der mündlichen Prüfung: 14. Januar 2000

Acknowledgements

First and foremost I would like to thank my my supervisor Prof. Dr. Dieter Sondermann who introduced me to the field of credit risk modelling, gave me valuable advice and support and who created the ideal environment for productive research at his department in Bonn. Amongst the many people who gave me opportunity to discuss my ideas and provided valuable feedback I would like to thank particularly Dr. Paul Wilmott, Dr. David Lando, Dr. Pierre Mella-Barral, Dr. Daniel Sommer, Dr. Dietmar Leisen, Antje Dudenhausen, David Epstein, Christopher Lotz, Lutz Schlögl, Gabi Zimmermann, Christian Zühlsdorff and my fellow students in the EDP. I would also like to thank the participants at the various conferences and seminars where I presented my work for their questions, comments and suggestions.

For financial support I would like to thank the Deutsche Forschungsgemeinschaft who supported me through the SFB 303 at the University of Bonn, and the DAAD who funded my year at the London School of Economics.

Last but not least I would like to thank my parents for being there for me at all times during my education and beyond.

Contents

Notation

Default-Free Term Structure of Interest Rates

Bond Prices

$B(t,T)$	default-free zero coupon bond price
$\mu(t,T)$	drift of zero coupon bond price $B(t,T)$
$\eta(t,T)$	volatility of zero coupon bond price $B(t,T)$
$\widehat{B}$	default-free coupon bond price

Default-Free Interest Rates

r	default-free instantaneous short rate (default-free short rate)
β	default-free discount factor: $\beta(t) = \exp\{-\int_0^t r(s)ds\}$
$f(t,T)$	default-free continuously compounded instantaneous forward rate (default-free forward rate)
$\alpha(t,T)$	drift of the default-free forward rate $f(t,T)$
$\sigma(t,T)$	volatility of the default-free forward rate $f(t,T)$
$a(t,T)$	integral of forward rate volatility: $a(t,T) = -\int_t^T \sigma(t,s)ds$
$F(t,T_1,T_2)$	simply compounded default-free forward rate over $[T_1,T_2]$

Defaultable Term Structure of Interest Rates

Defaultable Bond Prices

	defaultable quantities usually carry an $\overline{\text{overbar}}$
$\overline{B}(t,T)$	defaultable zero coupon bond price
$\overline{\mu}(t,T)$	drift of defaultable zero coupon bond price
$\overline{\eta}(t,T)$	volatility of defaultable zero coupon bond price
$\widetilde{B}(t,T)$	defaultable zero bond price adjusted for defaults before t $\widetilde{B}(t,T) = \overline{B}(t,T)/Q(t)$

Defaultable Interest Rates

$\overline{r}$	defaultable instantaneous short rate (defaultable short rate)
$\overline{\beta}$	defaultable discount factor: $\overline{\beta}(t) = \exp\{-\int_0^t \overline{r}(s)ds\}$
$\overline{f}(t,T)$	defaultable continuously compounded instantaneous forward rate (defaultable forward rate)
$\overline{\alpha}(t,T)$	drift of the defaultable forward rate
$\overline{\sigma}(t,T)$	volatility of the defaultable forward rate
$\overline{a}(t,T)$	integral of forward rate volatility: $\overline{a}(t,T) = -\int_t^T \overline{\sigma}(t,s)ds$
$\overline{F}(t,T_1,T_2)$	defaultable simply compounded forward rate over $[T_1,T_2]$

Spread

$h(t)$	instantaneous short spread (short spread)
$h(t,T)$	continuously compounded instantaneous forward rate spread (forward spread)
$\sigma^h(t,T)$	volatility of forward spread
$\alpha^h(t,T)$	drift of forward spread

Default Models

Default Time

N	point process whose first jump triggers the default
λ	intensity of N
τ_i	time of the i-th jump of N
τ	time of default ($\tau = \tau_1$)

Recovery Models

π	cash recovery model: recovery = π in cash
c	equivalent recovery model: recovery = c default -free bonds
q	fractional recovery model: payoff reduced by q at each default
Q	fractional recovery model: final payoff $Q(T) = \prod_{\tau_i \le T}(1 - q_{\tau_i})$

Probabilities

$P(t,T)$	probability of survival from t to T
$P^{\text{def}}(t,T)$	probability of default in $[t,T]$
$\widetilde{P}(t,T)$	pseudo survival probability: $\widetilde{P}(t,T) = \overline{B}(t,T)/B(t,T)$

The CIR Model

x_i	i-th factor ($i = 1, \ldots, n$)
w	factor weights of the default-free short rate: $r(t) = \sum_{i=1}^{n} w_i x_i(t)$
$\overline{w}$	factor weights of the intensity: $\lambda(t) = \sum_{i=1}^{n} \overline{w}_i x_i(t)$
η	weights of affine combination of noncentral chi-squared RVs

Chapter 1

Introduction

Although the valuation of defaultable securities with methods of continuous time finance goes back to the initial proposal of Black and Scholes (1973), this area of research has been largely ignored for a long time. Only recently credit risk modelling and credit risk management have received renewed attention, both by academics and practitioners alike.

The practitioner's interest has been renewed by several factors. First, the European monetary union and the full liberalisation of the European capital markets removed the segmentation of the European corporate bond market into national markets, increased liquidity and competition, and forced market participants to compare credit risk Europe-wide. Simultaneously, since European government bonds are now denominated in Euros, credit risk also became the key determinant of different prices in the European government bond markets.

Secondly, apart from these Euro-driven factors, historically low nominal interest rates all over the world have driven many investors to seek higher yields by accepting more credit risk. With the growing liquidity of the government bond markets these markets became very efficient and bond traders and investors looking for mis-priced securities had to turn towards the less liquid and less transparent corporate and high yield bond market. After the recent credit and currency crises in Asia and Russia the need arose to hedge and re-evaluate these investments.

Thirdly, banking supervisors have moved towards accepting internal risk measurement models as a basis for regulatory capital prescriptions. These models are already being accepted for market risks and have lead to a significant relaxation of capital adequacy regulation in these areas. It is expected that in the next years internal risk measurement models will also be accepted for the determination of adequate capital reserves for credit risks. This will generate significant advantages for those banks that have a credit risk model in place which is accepted by the regulating agencies.

Finally, driven by the need to hedge and manage credit risks in a flexible way, new derivative securities have been developed to fulfill this need. The pricing and management of these *credit derivatives* requires more flexible and sophisticated credit risk models.

From an academic point of view, with the advent of the market-based models the mathematical modelling of the pure interest-rate risk in the bond market is coming closer to a generally accepted benchmark (see e.g. Sandmann and Sondermann (1997), Miltersen, Sandmann and Sondermann (1997), Brace, Gatarek and Musiela (1997) or Jamshidian (1997)). Of the remaining risk components in the bond market, credit risk is the largest unresolved modelling problem.

Furthermore, the analysis of credit risk opens the door to new fields of research which interface with other areas like continuous-time corporate finance (see e.g. Leland (1994), Leland and Toft (1996) and Mella-Barral and Perraudin (1997)) or asymmetric information (see e.g. Duffie and Lando (1997)). This is part of a general move of mathematical finance away from traditional pricing and hedging into other areas, where they have not been applied before.

Finally, the valuation of credit derivatives changed the focus of many credit risk models. Instead of developing an pricing framework which yields the fair prices for defaultable bonds, now these bonds are to be taken as input to derive prices for more exotic derivative securities. Therefore models had to be developed that had this degree of flexibility.

There are two main approaches to credit risk modelling. The first approach goes back to the initial proposal of Black and Scholes (1973), where a defaultable security is regarded as a contingent claim on the value of the issuing firm's assets and is valued according to option pricing theory. In these models the firm's value is assumed to follow a diffusion process and default is modeled as the first time the firm's value hits a pre-specified boundary. Because of the continuity of the processes used, the time of default is a predictable stopping time. The payoff in default is usually a constant cash payment representing the proceeds from liquidating the firm (possibly after bankruptcy costs). The models of Merton (1974), Black and Cox (1976) Geske (1977) Longstaff and Schwartz (1993) and Das (1995) are representatives of this approach.

In a second approach, the direct reference to the firm's value is abandoned, and the time of default is modeled directly as a totally inaccessible stopping time with an intensity. This approach is followed by Artzner and Delbaen (1992; 1994), Jarrow and Turnbull (1995), Lando (1994; 1998), Jarrow, Lando and Turnbull (1997), Madan and Unal (1998), Flesaker et.al. (1994), Duffie and Singleton (1997; 1999), Duffie, Schroder and Skiadas (1994), Duffie and Huang (1996), and Duffie (1994). The main difference between these models is the way in which the recovery of a defaulted bond is modelled.

In the following we will describe these two modelling approaches in more detail.

1.1 Firm's Value Models

1.1.1 Modelling Approach

In firm's value models a fundamentalist's approach to valuing defaultable debt is taken: It is assumed that there is a fundamental process V, usually interpreted as the total value of the assets of the firm that has issued the bonds. The value of the firm V is assumed to change stochastically, usually it follows a lognormal diffusion process

$$dV = \mu V dt + \sigma V dW.$$

It is the driving force behind the dynamics of the prices of all securities issued by the firm, all claims on the firm's value are modelled as derivative securities with the firm's value as underlying.

Default can be triggered in two ways: Either V is only used to pay off the debt at the maturity of the contract. A default occurs at maturity if V is insufficient to pay back the outstanding debt but during the lifetime of the contract a default can not be triggered.

Alternatively (and more realistically) one can assume that a default is already triggered as soon as the value of the collateral V falls below a barrier $\bar{S}$. This feature is exactly identical to a standard knockout barrier in equity options.

1.1.2 Survey of the Literature

The firm's value approach is historically the oldest to the pricing of defaultable securities in modern continuous-time finance. It was first proposed by Black and Scholes in their article "The Pricing of Options *and Corporate Liabilities*" (1973) which already explicitly refers to corporate bond pricing in its title. Merton (1974) expands on this idea. In these models a default can only occur at maturity of the debt, the payoff of the firm's shares is like an European option on the firm's value.

In Black and Cox (1976) this approach is extended to allow for defaults before maturity of the debt if the firm's value hits a lower boundary. Now the model has more similarity with a barrier option model. Black and Cox show how to value a variety of corporate bonds and bond covenants in this framework. Further papers using this approach in a default-free interest rate setup are Merton (1977) and Geske (1977). Geske models defaultable debt as a compound option on the firm's value.

Recently Longstaff and Schwarz (1993) have managed to reach semi-closed form solutions (an infinite series) for defaultable bonds in a firm's value model with stochastic interest rates that can be correlated with the firm's value process. They use the Vasicek (1977) model for the default-free interest rates and have to assume a constant initial default-free term structure. Another, much simpler solution to this problem is given by Briys and de Varenne (1997) by modifying the approach such that a default is triggered when the T*-forward* price of the firm's value hits a lower barrier.

By now there are many extensions to the original modelling approach. Schönbucher (1996b) and later Zhou (1997) extend this approach to allow jumps in the process for V thus introducing a jump-diffusion process for the firm's value, which solves the problem of the unrealistically low short-term credit spreads that are found in all diffusion-based firm's value models. Schönbucher (1996b) also gives an efficient algorithm for the numerical solution of the pricing problem. In Epstein et.al. (1998) the firm's value model is modified such that not the firm's value but the firm's *cash flow* is the stochastic variable. The firm's value is then derived from accumulated cash flow, which makes the problem similar to an option pricing problem for Asian options.

The firm's value approach has also been implemented in a commercial model package which is marketed by KMV corporation. The KMV model is loosely based on the original Merton (1974) and Black-Scholes (1973) approach, but it draws its main strength from a judicious (but not model-consistent) use of a large database of historical defaults.

Because of their more explainative approach firm's value models have been popular in more theoretical areas, too. In an approach initiated by Leland (1994), Leland and Toft (1996) and Mella-Barral and Perraudin (1997) the firm's value framework is used to analyse strategic interaction between debtors and creditors. Duffie and Lando (1997) show in a setup with asymmetric information that there is a close link between firm's value models and the intensity models.

1.1.3 Advantages and Disadvantages

The firm's value models for defaultable bonds are well suited if the relationship between the prices of different securities issued by the firm is of importance, e.g. for convertible bonds or callable bonds that can be converted into shares when called by the issuer. Furthermore the model allows to price defaultable bonds directly from fundamentals, from the firm's value. Thus these models can give a fair price of a defaultable bond as output.

The foundation on fundamentals makes models of this type well-suited for the analysis of questions from corporate finance like the relative powers of shareholders and creditors or questions of optimal capital structure design.

This strength, the orientation towards fundamentals, is also one of the model's weaknesses: Often it is hard to define a meaningful process for the firm's value, let alone observe it continuously. It can be very hard to calibrate such a firm's value process to market prices, and for some issuers, like souvereign debt, it may not exist at all. Furthermore the model may very quickly become too complex to analyse in a real-world application. If one were to model the full set of claims on the value of the assets of a medium sized corporation one may very well have to price twenty or more classes of claims: from banks, shareholders and private creditors down to workers' wages, taxes and suppliers demands. This obviously becomes quickly unfeasible. On the other hand it seems that firm's value models are tailor-made for collateralised loans with traded collateral.

A second weakness of the firm's value models is the unrealistic nature of the short term credit spreads implied by the model. These spreads are very low and tend towards zero as the maturity of the debt approaches zero. This is immediately obvious from the fact that the time of default is a *predictable* stopping time. A predictable stopping time has zero intensity at all times except the time of default, and – as will be shown later on – the intensity determines the short-term credit spread.

Finally, for the pricing of credit risk derivatives one would like to have a model where the prices of defaultable bonds can be taken as *fundamentals* and do not have to be calculated (which then necessarily means a calibration process).

1.2 Intensity Models

In the intensity models the time of default is modeled directly as the time of the first jump of a Poisson process with random intensity (a Cox process), or – more generally – as a totally inaccessible stopping time with an intensity. In this group of models a striking similarity to default-free interest rate modelling is found.

The first models of this type were developed by Jarrow and Turnbull (1995), Madan and Unal (1998) and Duffie and Singleton (1997). Jarrow and Turnbull consider the simplest case where the default is driven by a Poisson process with constant intensity with known payoff at default. This is changed in the Madan and Unal model where the intensity of the default is driven by an underlying stochastic process that is interpreted as firm's value process, and the payoff in default is a random variable drawn at default, it is not predictable before default. Madan and Unal estimate the parameters of their process using rates for certificates of deposit in the Savings and Loan Industry.

Duffie and Singleton (1997) developed a similar model where the payoff in default is also cash, but denoted as a fraction $(1-q)$ of the value of the defaultable security just before default. This model was applied to a variety of problems including swap credit risk, estimation, and two-sided credit risk, by a group around Duffie (Duffie and Singleton (1997), Duffie, Schroder and Skiadas (1994), Duffie and Huang (1994) and Duffie (1994)).

Lando (1998) developed the Cox-process methodology with the iterated conditional expectations which will be used in the section on the pricing of credit derivatives later on. His model has a default payoff in terms a certain number of default-free bonds and he applies his results to a Markov chain model of credit ratings transitions.

In the Schönbucher (1996a; 1997a) model multiple defaults can occur and instead of liquidation with cash payoffs a restructuring with random recovery rate takes place. The model is set in a Heath- Jarrow- Morton framework and a rich variety of credit spread dynamics is allowed. For many pricing purposes the model can be reduced to a similar form of Duffie and Singleton, and in Schönbucher (1997b) it is applied to the pricing of several credit risk derivatives.

There is a variety of other models that fall into the class of intensity based models, we only mention Flesaker et.al. (1994), Artzner and Delbaen (1992; 1994) and Jarrow and Turnbull (1997). On the empirical side the papers by Duffee (1995), Duffie and Singleton (1997) and Düllmann et.al. (1999) have to be mentioned. In these papers the authors estimate the parameters for the stochastic process of the credit spread for the Duffie-Singleton model.

The intensity models have also been implemented in a commercial software package. The model is called *Credit Risk+* and it was developed by Credit Suisse Financial Products as a tool for the portfolio management of credit risk. In this model a default is triggered by the jump of a Poisson process whose intensity is randomly drawn for each debtor class.

1.3 Credit Rating Transition Models

The first continuous-time model of credit-risk pricing in a rating-transition framework is due to Lando (1994) and Jarrow, Lando and Turnbull (1997). This model only incorporates the Markov-chain dynamics of the ratings without allowing for stochastic spread dynamics within the rating classes.

Lando (1998) extends his model to have stochastic credit spreads by incorporating a stochastic multiplier in front of the transition generator matrix. This introduces some basic stochasticity into the credit spreads but still does not allow for fully general spread dynamics in all classes, because the credit spreads of all rating classes are driven by the same factor.

A different approach to incorporated stochastic credit spreads is taken by Das and Tufano (1994) who extend the Jarrow- Lando- Turnbull model to incorporate stochastic *recovery rates*. Thus they have stochastic credit spreads within the individual classes although the default intensities remain constant. Their model is set up as a discrete-time approximation to a continuous-time model. Although this model achieves the aim of generating stochastic credit spreads it still has some important shortcomings. It does not seem plausible why the likelihood of default should remain constant while the expected recovery changes (typically the converse is the case) and the set of possible credit spreads is bounded from above by the credit spreads for zero recovery.

In this thesis the credit rating transition models will be extended to a Heath- Jarrow-Morton model which can incorporate fully stochastic dynamics for the credit spreads of all credit classes. The term structures of credit spreads of all rating classes will be modelled and joined with a default model in a consistent and arbitrage-free way.

1.4 Credit Derivatives Literature

Despite a number of articles that have been written on the application and uses of credit derivatives, there is very little literature on the direct *pricing* of credit derivatives. Among the exceptions here are the articles of Duffie (1999) and Longstaff and Schwartz (1995). Das (1998) gives a simple discretisation of the HJM- approach to credit spreads, and Pierides (1997) and Das (1995) use a firm's value approach to value credit derivatives. Basket default swaps are the topic of Duffie (1998) and Li (1999).

Good but not very rigorous introductions to the applications and uses of credit derivatives are the books by Mathieu and d'Herouville (1998), Tavakoli (1998), Das (1998) and Nelken (1999).

1.5 Structure of the Thesis

The rest of the thesis is organized as follows:

The next chapter covers the term-structure modelling of defaultable bonds. The model proposed there is the adaptation of the Heath- Jarrow- Morton (HJM) model to the case of a term structure of defaultable bond prices. The arbitrage-free dynamics of the defaultable bond prices are derived and a new default model – the *multiple default model* – is introduced. The model is extended to a model with jumps in the defaultable forward rates and then to a rating transition model with stochastic term structures of credit spreads in all ratings classes. All these extensions flow naturally from the original HJM forward-rate approach.

In the following chapter, the pricing of credit derivatives is discussed. After a short introduction to the most popular types of credit derivatives and their applications, closed-form pricing formulae are given for the most important credit derivatives. These pricing formulae are derived in different recovery model frameworks (notably equivalent recovery and fractional recovery), and for a Gaussian HJM and a Cox-Ingersoll-Ross specification of the interest-rate and credit spread dynamics. The application of these different modelling approaches to concrete pricing problems allows us to identify the relative advantages and disadvantages of these models. The appendices contain some more technical calculations that are necessary for the derivation of the pricing formulae for the credit derivatives in the two model specifications.

In the final sections of each chapter the main results of the chapter are summarised and areas for future research are pointed out. A more detailed description of the contents of the chapters can be found in the first section of each chapter.

Chapter 2

Term Structure Modelling of Defaultable Bonds

In this chapter[1] we present a model of the development of the term structure of defaultable interest rates that is based on a multiple-defaults model. Instead of modelling a cash payoff in default we assume that defaulted debt is restructured and continues to be traded.
We use the Heath-Jarrow-Morton (HJM) (1992) approach to represent the term structure of defaultable bond prices in terms of forward rates. The focus of the chapter lies on the modelling the development of this term structure of defaultable bond prices and we give conditions under which these dynamics are arbitrage-free. These conditions are a drift restriction that is closely related to the HJM drift restriction for default-free bonds, and the restriction that the defaultable short rate must always be not below the default-free short rate.
Similar restrictions are derived for two extensions of the model setup, the first one is in a marked point process framework and allows for jumps in the defaultable forward rates at times of default, and the second one is a general ratings transition framework which can incorporate stochastic dynamics for the credit spreads in all ratings classes and also stochastic transition intensities.

2.1 Introduction

Most bankruptcy codes provide several alternative procedures to deal with defaulted debt and the debtors. The most obvious option is to liquidate the debtor's remaining assets

[1]This first parts of this chapter are based upon Schönbucher (1998).

and distribute the proceeds amongst the creditors, but a often more popular alternative is to reorganize the defaulted issuer and keep the issuer in operation. The latter alternative has the advantage of preserving the value of the debtor's business as a going concern and it avoids inefficient liquidation sales. Frequently there is no alternative to reorganisation, either because a liquidation of an issuer is impossible (e.g. for sovereign debtors) or because it is undesirable (if a liquidation would have a large macroeconomic effect).

In their empirical study Franks and Torous (1994) found the following:
A default of a bond does not mean that this bond becomes worthless, usually there is a positive recovery rate between 40 and 80 percent. This recovery rate varies significantly between firms.
The majority of firms in financial distress are reorganized and re-floated, they are not liquidated.
On average, most of the compensation payments (about two thirds) are in terms of securities of the reorganized firm, not in cash.
If a firm is reorganized, and the debtors are paid in terms of newly issued debt, then a second default of this firm on its (newly issued) debt is possible. In principle there could be a sequence of any number of defaults each with a subsequent restructuring of the defaulted firm's debt.

In this chapter we present a model of defaultable bond prices in which a defaulted issuer is not liquidated but reorganized at default. Multiple defaults can occur and the magnitude of the losses in default is not predictable.

We use the Heath-Jarrow-Morton (HJM) (1992) framework to represent the term structure of defaultable bond prices in terms of forward rates and give conditions under which these dynamics are arbitrage-free. These conditions are a drift restriction that is closely related to the HJM drift restriction for default-free bonds, and the restriction that the defaultable short rate must always be not below the default-free short rate.

The model in this chapter is based on the intensity-approach, an approach in which the time of default is a totally inaccessible stopping time which has an intensity process. Amongst others this approach is followed by Artzner and Delbaen (1992; 1994), Jarrow and Turnbull (1995), Lando (1994; 1998), Jarrow, Lando and Turnbull (1997), Madan and Unal (1998), Flesaker et.al. (1994), Duffie and Singleton (1997; 1999), Duffie, Schroder and Skiadas (1994), Duffie and Huang (1996), and Duffie (1994). In all these models a cash payoff (or a payoff in default-free bonds) is specified in default. Therefore only one default is allowed, and after default the firm that had issued the debt is liquidated. This excludes reorganisation of defaulted debt as well as multiple defaults. Usually (except Madan and Unal (1998)) the magnitude of the payoff in default is predictable, too. In this model there is the possibility of multiple defaults and the magnitude of the recovery need not be predictable. It should furthermore be pointed out that many of the results (e.g. the HJM drift restriction on the defaultable bond prices or the relationship between the short credit spread and the default intensity) are not restricted to the intensity-based framework, but remain valid also for other default models.

We start with a model in which the value of a defaultable bond drops to zero upon default. While this case has already been extensively studied in the literature it is a good introduction to more general models. After deriving the interconnections between the dynamics of the defaultable interest rates and the defaultable bond prices, we derive the key relationship that *under the martingale measure the difference between the defaultable short rate and the default-free short rate is the intensity of the default process* with an argument using the savings accounts. This result drives the conditions for the absence of arbitrage that are derived subsequently, and the arbitrage-free dynamics of the defaultable bond prices. We find a very strong similarity between the defaultable and the default free interest rate dynamics and drift restrictions, as both have to satisfy the HJM drift restrictions. (Similar results have been shown by Duffie in (1994) and Lando (1998).)
Next we explore how a model of the spread of the defaultable forward rates over the default-free forward rates may be used to add a default-risk module to an existing model of default-free interest rates. Surprisingly, for forward rates this spread can be negative although there has to be a positive spread for the short rates. An example is given demonstrating that this is only possible under strong correlation between spreads and default-free interest rates.

In the following sections we propose a model that includes positive recovery rates, reorganisations of the defaulted firms with the possibility of multiple defaults and uncertainty about the magnitude of the default. Even though it may seem that this will make the model far more complicated the restrictions for absence of arbitrage and the price dynamics remain unchanged. This model is closely related to the fractional recovery model proposed by Duffie and Singleton (1994; 1997; 1999).

Instead of modelling the defaultable forward rates, previous intensity models concentrated on modelling the default intensity (i.e. the short credit spread) directly. These two are connected and the link is shown in the next section. We show that the results of the classical intensity models can be recovered from the defaultable forward model directly. Particularly the representation of the defaultable bond prices as expectation of a defaultable discount factor follows much more easily than in most intensity models. It is also shown that the only restriction for absence of arbitrage is to ensure a positive spread between defaultable and default-free short rate.

In the following section the defaultable forward rates (and thus the bond prices) are allowed to change discontinously at default times if there are multiple defaults. Here we use the methods of Björk, Kabanov and Runggaldier (BKR) (1996) and give the drift- and no-arbitrage conditions for this more general version of the model.

The chapter is concluded with another extension of the model which incorporates ratings transitions. In this section we model the defaultable forward rates for all rating classes simultaneously and analyse the conditions that ensure absence of arbitrage for a given ratings transition intensity matrix and a given volatility structure of the credit spreads within each ratings class. Again, these conditions turn out to be closely related to the classical HJM conditions which makes the model accessible to a numerical implementation. Instead of interpreting the model as a ratings transition model it can also be viewed as a model including a *credit crisis* with stochastic transition from the 'normal' state to the 'crisis' state which is characterised by much higher spreads, volatility and default risk. A regime shift of this kind can not be replicated in most other credit risk models, but it is a very important feature of real debt markets.

2.2 Setup and Notation

For ease of exposition we first introduce the simplest setup which will be generalised in the following sections to include positive recovery rates, multiple defaults and jumps in the defaultable term structure.

The model is set in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{(t\geq 0)}, P)$ where P is some subjective probability measure. We assume the filtration $(\mathcal{F}_t)_{(t\geq 0)}$ satisfies the usual conditions[2] and the initial filtration $\mathcal{F}_0$ is trivial. We also assume a finite time horizon $\bar{T}$ with $\mathcal{F} = \mathcal{F}_{\bar{T}}$, all definitions and statements are understood to be only valid until this time horizon $\bar{T}$.

The time of default is defined as follows:

[2]See Jacod and Shiryaev (1988).

Definition 2.1

The time of default *is a stopping time* τ*. We denote with* $N(t) := \mathbf{1}_{\{\tau \leq t\}}$ *the* default indicator function *and* $A(t)$ *the predictable compensator of* $N(t)$*, thus*

$$M(t) := N(t) - A(t)$$

is a (purely discontinuous) martingale. A *is nondecreasing (because* N *is), predictable and of finite variation. Frequently we will assume that* N *has an intensity* $\lambda(s)$ *which means that* A *can be represented as*

$$A(t) = \int_0^t \lambda(s)\, ds. \tag{2.1}$$

The filtration $(\mathcal{F}_t)_{(t\geq 0)}$ is generated[3] by n independent Brownian motions W^i, $i = 1, \dots, n$ and the default indicator $N(t)$.

For the default-free bond markets we use the HJM setup:

Definition 2.2

1. *At any time* t *there are* default-risk free zero coupon bonds *of all maturities* $T > t$*. The time-*t *price of the bond with maturity* T *is denoted by* $B(t,T)$*.*

2. *The* continuously compounded default-free forward rate *over the period* $[T_1, T_2]$ *contracted at time* t *is defined (for* $T_2 > T_1 \geq t$*)*

$$f(t, T_1, T_2) = \frac{1}{T_2 - T_1} \left(\ln B(t, T_1) - \ln B(t, T_2) \right). \tag{2.2}$$

3. *If the* T*-derivative of* $B(t,T)$ *exists, the* continuously compounded instantaneous default-free forward rate *at time* t *for date* $T > t$ *is defined as*

$$f(t,T) = -\frac{\partial}{\partial T} \ln B(t,T). \tag{2.3}$$

4. *The* instantaneous default-free short rate $r(t)$*,*
the default-free discount factor $\beta(t)$
and the default-free bank account $b(t)$ *are defined by*

$$r(t) := f(t,t), \qquad \beta(t) := \exp\{-\int_0^t r(s)ds\}, \quad b(t) := 1/\beta(t). \tag{2.4}$$

[3]This assumption will be relaxed later on to include a marked point process in the case of multiple defaults.

We use similar notation to describe the term structure of the defaultable bonds:

Definition 2.3

1. *At any time t there are* defaultable zero coupon bonds *of all maturities T (where $T > t$). The time-t price of the bond with maturity T is denoted by $\overline{B}(t,T)$. The payoff at time T of this bond is $\mathbf{1}_{\{\tau>T\}} = 1 - N(t)$: one unit of account if the default has not occurred until T, and nothing otherwise.*

2. *The* continuously compounded defaultable forward rate *over the period $[T_1, T_2]$ contracted at time t is defined (for $T_2 > T_1 \geq t$)*

$$\overline{f}(t, T_1, T_2) = \frac{1}{T_2 - T_1}\left(\ln \overline{B}(t, T_1) - \ln \overline{B}(t, T_2)\right). \tag{2.5}$$

3. *If the T-derivative of $\overline{B}(t,T)$ exists, the* continuously compounded instantaneous defaultable forward rate *at time t for date $T > t$ is defined as*

$$\overline{f}(t,T) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T). \tag{2.6}$$

4. *The* instantaneous defaultable short rate $\overline{r}(t)$,
the defaultable discount factor $\overline{\beta}(t)$
and the defaultable bank account $c(t)$ *are defined by*

$$\overline{r}(t) := \overline{f}(t,t), \quad \overline{\beta}(t) := \exp\{-\int_0^t \overline{r}(s)ds\}, \quad c(t) := \mathbf{1}_{\{t<\tau\}}\frac{1}{\overline{\beta}(t)}. \tag{2.7}$$

All definitions of defaultable interest rates are only valid for times $t < \tau$ before default.

To shorten notation the reference to the continuous compounding frequency of the interest rates or yields is often omitted, and we only refer to default-free and defaultable *forward rates* (=continuously compounded instantaneous forward rates) and *short rates* (=instantaneous short rates). We will also often use *default-free* in place of *default-free* to denote non-defaultable quantities.

The defaultable forward rate $\overline{f}(t, T_1, T_2)$ as it is defined above is *not* the value of a T_1-forward contract on a defaultable bond with maturity T_2, but the *promised* yield of the following portfolio:

short	one	defaultable bond	$\overline{B}(t, T_1)$
long	$\overline{B}(t, T_1)/\overline{B}(t, T_2)$	defaultable bonds	$\overline{B}(t, T_2)$.

A forward contract on the defaultable bond T_2 would involve a short position in the default free bond $B(t, T_1)$. See also section 2.4.2 for some consequences of this definition.

The defaultable bank account $c(t)$ is the value of \$ 1 invested at $t = 0$ in a defaultable zero coupon bond of very short maturity and rolled over until t, given there has been no default until t. It will play a similar role to the default-free bank account $b(t)$ in default-free interest rate modelling.

In the definition of the defaultable forward rates — to avoid taking logarithms of defaultable bond prices that are zero — we assume that a future default cannot be predicted with certainty. At any time $t < \tau$ strictly before default, and for every finite prediction-horizon T ($t < T < \infty$) the probability of a default until T is not one: $P[\tau \le T | \mathcal{F}_t] < 1$. This can be achieved by setting the default time to be the first time τ' at which a future default can be predicted with certainty: $\tau' := \inf\{t \ge 0 | \exists\, T < \infty \text{ s.t. } P[\tau \le T | \mathcal{F}_t] = 1\}$, effectively moving the time of default forward in time[4].

We assume τ has been defined as above. This assumption is in keeping with the real-world legal provisions that a bankruptcy must be filed as soon as the fact of the bankruptcy is known. Furthermore it does not change any of the qualitative features of the model. In addition to this we assume that all (forward) interest rates have continuous paths and that the instantaneous forward rates are well-defined.

2.3 Pricing with Zero Recovery

2.3.1 Dynamics: The defaultable Forward Rates

Given the above definitions we can start to explore the connections between the dynamics of the defaultable bond prices and the defaultable forward rates. We assume the following representation as stochastic integrals for the dynamics of the defaultable forward rates $\overline{f}(t,T)$ and the defaultable bonds $\overline{B}(t,T)$:

Assumption 2.1

1. The dynamics of the defaultable forward rates are given by

$$d\overline{f}(t,T) = \overline{\alpha}(t,T)\; dt + \sum_{i=1}^{n} \overline{\sigma}_i(t,T)\; dW^i(t). \tag{2.8}$$

[4] By definition $\tau' \le \tau$, but $\tau' = \infty$ is possible.

2. The dynamics of the defaultable bond prices are [5]

$$\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = \overline{\mu}(t,T)dt + \sum_{i=1}^{n} \overline{\eta}_i(t,T)\, dW^i(t) - dN(t). \tag{2.9}$$

3. The integrands $\overline{\alpha}(t,T), \overline{\sigma}_i(t,T), \overline{\mu}(t,T)$ and $\overline{\eta}_i(t,T)$ are predictable processes that are regular enough to allow
– differentiation under the integral sign
– interchange of the order of integration
– partial derivatives with respect to the T-variable
– bounded prices $\overline{B}(t,\cdot)$ for almost all $\omega \in \Omega$.

We start by analysing the consequences of the specification (2.8) of the defaultable forward rates. The *dynamics of the defaultable spot rate process* are [6]

$$\begin{aligned} \overline{r}(t) &= \overline{f}(t,t) = \overline{f}(0,t) + \int_0^t \overline{\alpha}(s,t)ds \\ &+ \sum_{i=1}^{n} \int_0^t \overline{\sigma}_i(s,t)dW^i(s). \end{aligned} \tag{2.10}$$

From definition (2.6) of the defaultable forward rates and definition 2.3 of the defaultable bonds the price of a defaultable zero coupon bond is given by

$$\overline{B}(t,T) = (1 - N(t)) \exp\left\{ - \int_t^T \overline{f}(t,s)\, ds \right\}. \tag{2.11}$$

The factor of $(1 - N(t))$ follows from the default condition $\overline{B}(t,T) = 0$ for $t \geq \tau$. Writing $G(t,T) := \int_t^T \overline{f}(t,s)\, ds$ this yields for $t \leq \tau$ using Itô's lemma

$$d\overline{B}(t,T)/\overline{B}(t-,T) = -dG(t,T) + \frac{1}{2} d < G, G > -dN, \tag{2.12}$$

where we have used that G is continuous. For the process $G(t,T)$ we have (see HJM (1992))

$$G(t,T) - G(0,T) = \int_t^T \left[\overline{f}(t,s) - \overline{f}(0,s)\right]\, ds - \int_0^t \overline{f}(0,s)\, ds$$

[5] The notation $dY(t)/Y(t-) = dX(t)$ is a shorthand for $dY(t)/Y(t-) = dX(t)$ for $Y(t-) > 0$ and $dY(t) = 0$ for $Y(t-) = 0$.

[6] It is understood that dynamics of defaultable interest rates are always the dynamics before default $t < \tau$.

$$
\begin{aligned}
&= \int_t^T \int_0^t \overline{\alpha}(u,s)\, du\, ds + \sum_{i=1}^n \int_t^T \int_0^t \overline{\sigma}_i(u,s)\, dW^i(u)\, ds \\
&\quad - \int_0^t \overline{f}(0,s)\, ds \\
&= \int_0^t \int_t^T \overline{\alpha}(u,s)\, ds\, du + \sum_{i=1}^n \int_0^t \int_t^T \overline{\sigma}_i(u,s)\, ds\, dW^i(u) \\
&\quad - \int_0^t \overline{f}(0,s)\, ds \\
&= \int_0^t \int_u^T \overline{\alpha}(u,s)\, ds\, du + \sum_{i=1}^n \int_0^t \int_u^T \overline{\sigma}_i(u,s)\, ds\, dW^i(u) \\
&\quad - \int_0^t \overline{f}(0,s)\, ds - \int_0^t \int_u^t \overline{\alpha}(u,s)\, ds\, du \\
&\quad - \sum_{i=1}^n \int_0^t \int_u^t \overline{\sigma}_i(u,s)\, ds\, dW^i(u) \\
&= \int_0^t \overline{\gamma}(u,T)\, du - \sum_{i=1}^n \int_0^t \overline{a}_i(u,T)\, dW^i(u) \\
&\quad - \int_0^t \overline{f}(0,s)\, ds - \int_0^t \int_0^s \overline{\alpha}(u,s)\, du\, ds \\
&\quad - \sum_{i=1}^n \int_0^t \int_0^s \overline{\sigma}_i(u,s)\, dW^i(u)\, ds \\
&= \int_0^t \left(\overline{\gamma}(u,T) - \overline{r}(u)\right)\, du - \sum_{i=1}^n \int_0^t \overline{a}_i(u,T)\, dW^i(u)
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{a}_i(t,T) &:= -\int_t^T \overline{\sigma}_i(t,v)\, dv && (2.13) \\
\overline{\gamma}(t,T) &:= \int_t^T \overline{\alpha}(t,v)\, dv. && (2.14)
\end{aligned}
$$

The main tool in the equations above is Fubini's theorem and Fubini's theorem for stochastic integrals (see e.g. HJM (1992) and Protter (1990)).

With this result we reach the dynamics of the defaultable zero coupon bond prices

$$
\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = \left[-\overline{\gamma}(t,T) + \overline{r}(t) + \frac{1}{2}\sum_{i=1}^n \overline{a}_i^2(t,T) \right] dt
$$

$$+\sum_{i=1}^{n} \overline{a}_i(t,T)\, dW^i(t) - dN(t). \tag{2.15}$$

The final condition $\overline{B}(T,T) = 0$ for $\tau < T$ is automatically satisfied by the functional specification of $\overline{B}$.

The above derivation of the dynamics of $G(t,T)$ follows the derivation of the dynamics of the default-free bond prices in HJM (1992). Here the only addition is the jump term $-dN(t)$ which is introduced by the default process. Summing up:

Proposition 2.1

1. Given the dynamics of the defaultable forward rates (2.8)

(i) the dynamics of the defaultable bond prices are given by

$$\begin{aligned}\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} &= \left[-\overline{\gamma}(t,T) + \overline{r}(t) + \frac{1}{2}\sum_{i=1}^{n} \overline{a}_i^2(t,T)\right] dt \\ &\quad + \sum_{i=1}^{n} \overline{a}_i(t,T)\, dW^i(t) - dN(t).\end{aligned} \tag{2.16}$$

where $\overline{a}_i(t,T)$ and $\overline{\gamma}(t,T)$ are defined by (2.13) and (2.14) resp..

(ii) the dynamics of the defaultable short rate are given by

$$\begin{aligned}\overline{r}(t) &= \overline{f}(t,t) = \overline{f}(0,t) + \int_0^t \overline{\alpha}(s,t)ds \\ &+ \sum_{i=1}^{n} \int_0^t \overline{\sigma}_i(s,t)dW^i(s).\end{aligned} \tag{2.17}$$

2. Given the dynamics (2.9) of the defaultable bond prices the dynamics of the defaultable forward rates are (for $t \le \tau$) given by (2.8) with

$$\overline{\alpha}(t,T) = \sum_{i=1}^{n} \overline{\eta}_i(t,T)\frac{\partial}{\partial T}\overline{\eta}_i(t,T) - \frac{\partial}{\partial T}\overline{\mu}(t,T) \tag{2.18}$$

$$\overline{\sigma}_i(t,T) = -\frac{\partial}{\partial T}\overline{\eta}_i(t,T). \tag{2.19}$$

Proof: 1.) has been derived above, 2.) follows from Itô's lemma on $\ln \overline{B}(t,T)$ and taking the partial derivative w.r.t. T.

□

These relationships are well-known in the case of the default-risk free term structure. Assume the following dynamics of the default-free forward rates $f(t,T)$ and the default-free bond prices $B(t,T)$:

Assumption 2.2

1. The dynamics of the default risk free forward rates are given by

$$df(t,T) = \alpha(t,T)\,dt + \sum_{i=1}^{n} \sigma_i(t,T)\,dW^i(t). \tag{2.20}$$

2. The dynamics of the default risk free bond prices are

$$\frac{dB(t,T)}{B(t-,T)} = \mu(t,T)dt + \sum_{i=1}^{n} \eta_i(t,T)\,dW^i(t). \tag{2.21}$$

3. The integrands $\alpha(t,T), \sigma_i(t,T), \mu(t,T)$ and $\eta_i(t,T)$ are predictable processes that are regular enough to allow

- *differentiation under the integral sign*
- *interchange of the order of integration*
- *partial derivatives with respect to the T-variable*
- *bounded prices $B(t,\cdot)$ for almost all $\omega \in \Omega$.*

The dynamics of the default-free term structure do not contain any jumps at τ. Volatilities and drifts may change at τ but the direct impact of the default is only on the defaultable bonds.

Given these dynamics the following proposition is a well-known result by Heath, Jarrow and Morton (1992).

Proposition 2.2

1. Given the dynamics of the risk free forward rates (2.20)

(i) the dynamics of the risk free bond prices are given by

$$\begin{aligned} \frac{dB(t,T)}{B(t-,T)} &= \left[-\gamma(t,T) + r(t) + \frac{1}{2}\sum_{i=1}^{n} {a_i}^2(t,T)\right] dt \\ &\quad + \sum_{i=1}^{n} a_i(t,T)\,dW^i(t). \end{aligned} \tag{2.22}$$

where $a_i(t,T)$ *and* $\gamma(t,T)$ *are defined by*

$$a_i(t,T) := -\int_t^T \sigma_i(t,v)\, dv \tag{2.23}$$

$$\gamma(t,T) := \int_t^T \alpha(t,v)\, dv. \tag{2.24}$$

(ii) the dynamics of the risk free short rate are given by

$$\begin{aligned} r(t) &= f(t,t) = f(0,t) + \int_0^t \alpha(s,t)ds \\ &+ \sum_{i=1}^n \int_0^t \sigma_i(s,t)dW^i(s). \end{aligned} \tag{2.25}$$

2. Given the dynamics (2.21) of the risk free bond prices the dynamics of the defaultable forward rates are given by (2.20) with

$$\alpha(t,T) = \sum_{i=1}^n \eta_i(t,T)\frac{\partial}{\partial T}\eta_i(t,T) - \frac{\partial}{\partial T}\mu(t,T) \tag{2.26}$$

$$\sigma_i(t,T) = -\frac{\partial}{\partial T}\eta_i(t,T). \tag{2.27}$$

2.3.2 Change of Measure

Now that the connections between the dynamics of the defaultable zero coupon bonds and the forward rates are clarified, we can start analysing the conditions for absence of arbitrage opportunities in this model. We use the following standard definition:

Definition 2.4
There are no arbitrage opportunities if and only if there is a probability measure Q equivalent to P under which the discounted security price processes become local martingales. This measure Q is called the martingale measure, *and for any security price process $X(t)$ the discounted price process is defined as $\beta(t)X(t)$.*

The main tool to classify all to P equivalent probability measures is the following version of Girsanov's Theorem (see Jacod and Shiryaev (1988) III.3 and III.5 and BKR (1996)):

Theorem 2.3
Assume that the default process has an intensity. Let θ be a n-dimensional predictable

processes $\theta_1(t), \dots, \theta_n(t)$ and $\phi(t)$ a strictly positive predictable process with

$$\int_0^t \|\theta(s)\|^2 ds < \infty, \qquad \int_0^t |\phi(s)|\lambda(s) ds < \infty$$

for finite t. Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \sum_{i=1}^{n} \theta_i(t) dW^i(t) + (\phi(t) - 1) dM(t).$$

Assume that $\mathbf{E}[\, L(t)\,] = 1$ for finite t.
Then there is a probability measure Q equivalent to P with

$$dQ = L_{\bar{T}} dP \qquad \text{and} \qquad dQ_t = L_t dP_t \tag{2.28}$$

where $Q_t := Q|\, \mathcal{F}_t$ and $P_t := P|\, \mathcal{F}_t$ are the restrictions of Q and P on $\mathcal{F}_t$, such that

$$dW(t) - \theta(t)dt = d\widetilde{W}(t) \tag{2.29}$$

defines $\widetilde{W}$ as Q-Brownian motion and

$$\lambda_Q(t) = \phi(t)\lambda(t) \tag{2.30}$$

is the intensity of the default indicator process under Q.
Furthermore every probability measure that is equivalent to P can be represented in the way given above.

In the financial context here the processes θ_i are the *market prices of diffusion risk*, and the process ϕ represents a *market premium on jump risk* (in terms of a multiplicative factor per unit of jump intensity). To ensure absence of arbitrage the financial requirement of a well-defined set of market prices of risk with validity for all securities translates into the mathematical requirement of having a well-defined intensity process for the change of measure.

Given the defaultable bond price dynamics (2.9) the change of measure to the martingale measure leaves the volatilities of the defaultable bond prices unaffected, the same is true of the integral with respect to dN (the *compensator* of this integral has changed, though), the only effect is a change of drift in the defaultable bond price process.

From now on we will assume that the change of measure to the martingale measure has already been performed. The results of the preceding section on the dynamics remain valid if the underlying measure is the martingale measure. Therefore we simplify notation such that all specifications in section 2.3.1 are already with respect to Q. [7]

2.3.3 Absence of Arbitrage

By Itô's lemma we require under the martingale measure for absence of arbitrage that for all $t > T$

$$\mathbf{E}\left[\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)}\right] = r(t)\,dt. \tag{2.31}$$

This means using (2.16)

$$\begin{aligned} r(t)\,dt &= \mathbf{E}\left[\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)}\right] \\ &= \mathbf{E}\left[\left[-\overline{\gamma}(t,T)+\overline{r}(t)+\frac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)\right]dt\right] \\ &\quad +\mathbf{E}\left[\sum_{i=1}^{n}\overline{a}_i(t,T)\,dW^i(t) - dN(t)\right] \\ r(t) &= -\overline{\gamma}(t,T)+\overline{r}(t)+\frac{1}{2}\sum_{i=1}^{n} a_i^2(t,T)-\lambda(t) \end{aligned} \tag{2.32}$$

Now we have to take a closer look at the compensator $A(t)$ of the default indicator process $N(t)$. $A(t)$ is increasing and we assumed that $A(t)$ is also continuous, therefore $A(t)$ is differentiable almost everywhere on $\mathbb{R}^+$ and thus $N(t)$ has an intensity $dA(t) = \lambda(t)dt$. Then

$$-M(t) = -N(t) + A(t) = -N(t) + \int_0^t \lambda(s)\,ds \tag{2.33}$$

is a martingale by the definition of the predictable compensator. Now consider the value process of the *defaultable* bank account $c(t)$, i.e. the development of \$ 1 invested at time 0 at the defaultable short rate and rolled over from then on. By definition its value at time t is

$$c(t) = \mathbf{1}_{\{\tau>t\}}\exp\{\int_0^t \overline{r}(s)ds\}. \tag{2.34}$$

[7] If $P = Q$ then $\theta \equiv 0$ and $\phi \equiv 1$.

Under the martingale measure the discounted (discounting with the *default-free* interest rate) value process of c

$$\bar{c}(t) := \frac{c(t)}{b(t)} = \mathbf{1}_{\{\tau>t\}} \exp\{\int_0^t \overline{r}(s) - r(s)\, ds\} \tag{2.35}$$

must be a martingale. This is the Doleans-Dade exponential of

$$\hat{M}(t) := -N(t) + \int_0^{t\wedge\tau} \overline{r}(s) - r(s)\, ds, \tag{2.36}$$

which in turn must also be a martingale. (The martingale property can also be seen from $\hat{M}(t) = \int_0^t \frac{1}{\bar{c}(s-)} d\bar{c}(s)$ and the uniqueness of the Doleans-Dade exponential up to τ.) We use the freedom we had in the specification of $\overline{r}(t)$ for $t \geq \tau$ and set $\overline{r}(t) := r(t)$ for $t \geq \tau$.

Taking the difference of (2.33) and (2.36)

$$M(t) - \hat{M}(t) = \int_0^t \lambda(s) - \overline{r}(s) + r(s)\, ds \tag{2.37}$$

one sees that – while the l.h.s. is a martingale – the r.h.s. is predictable, the only predictable martingales are constant, thus we have for (almost) all s

$$\lambda(s) = \overline{r}(s) - r(s). \tag{2.38}$$

The hazard rate $\lambda(s)$ of the default is exactly the short interest rate spread. Note that this relationship can also be inverted to define the defaultable short rate as $\overline{r}(s) := r(s) + \lambda(s)$.

Equation (2.38) is the key relation that yields, substituted in (2.32), as necessary condition for the absence of arbitrage:

$$\overline{\gamma}(t,T) = \frac{1}{2}\sum_{i=1}^{n} a_i^2(t,T). \tag{2.39}$$

Substituting the definition of $\overline{\gamma}$ in this condition yields the results of the following theorem.

Theorem 2.4
The following are equivalent:

1. *The measure under which the dynamics are specified is a martingale measure.*

2. *(i) The short interest rate spread is the intensity of the default process. It is nonnegative.*

$$\lambda(t) = \overline{r}(t) - r(t) \geq 0. \tag{2.40}$$

(ii) The drift coefficients of the defaultable forward rates satisfy for all $t \leq T$, $t < \tau$

$$\int_t^T \overline{\alpha}(t,v)dv = \frac{1}{2}\sum_{i=1}^n \left(\int_t^T \overline{\sigma}_i(t,v)dv\right)^2 \tag{2.41}$$

or, differentiated,

$$\overline{\alpha}(t,T) = \sum_{i=1}^n \overline{\sigma}_i(t,T)\int_t^T \overline{\sigma}_i(t,v)dv. \tag{2.42}$$

(iii) The drift coefficients of the default-free forward rates satisfy for all $t \leq T$

$$\alpha(t,T) = \sum_{i=1}^n \sigma_i(t,T)\int_t^T \sigma_i(t,v)dv. \tag{2.43}$$

3. *(i)* $\overline{r}(t) - r(t) = \lambda(t) > 0.$

(ii) The dynamics of the defaultable bond prices are given by

$$\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = \overline{r}(t)dt + \sum_{i=1}^n \overline{a}_i(t,T)dW^i(t) - dN(t) \tag{2.44}$$

or, solving the s.d.e.

$$\begin{aligned}\overline{B}(t,T) \quad &= \quad \mathbf{1}_{\{\tau>t\}}\overline{B}(0,T)\exp\left\{\int_0^t \overline{r}(s)\,ds - \frac{1}{2}\sum_{i=1}^n \int_0^t a_i^2(s,T)\,ds\right.\\ &\qquad \left. + \sum_{i=1}^n \int_0^t \overline{a}_i(s,T)\,dW^i(s)\right\}.\end{aligned} \tag{2.45}$$

(iii) The dynamics of the default-free bonds satisfy under the martingale measure

$$\frac{dB(t,T)}{B(t-,T)} = r(t)dt + \sum_{i=1}^n a_i(t,T)dW^i(t). \tag{2.46}$$

Proof: 1.) ⇒ 2.) : (i) and (ii) have been derived above, (iii) has been shown in HJM (1992).
2.) ⇒ 3.) : 2.(i) and 3.(i) coincide, 3.(ii) follows from 2.(ii) and (i) by substituting in proposition 1, again (iii) is by HJM (1992).

3.) $\Rightarrow$ 1.): follows from the definition of the martingale measure.

□

The most important result of this section is equation (2.42), the defaultable-bond equivalent of the well-known Heath-Jarrow-Morton drift-restriction. This restriction has been derived for default-free bonds in HJM (1992), and, as we see here, it is also an important part of the modelling of the defaultable bonds' dynamics. This was first noted by Duffie (1994) and Lando (1998).

Another important insight is that precise knowledge of the nature of the default process N and its compensator A is not necessary for setting up an arbitrage-free model of the term structure of defaultable bonds. With the restrictions 2.) of theorem 2.4 one can set up a model of defaultable bonds that uses readily observable market data (the term structure of the defaultable forward rates) as input, without having to try and find out about the precise nature of N.

We assumed that the default process has an intensity: $dA(t) = \lambda(t)dt$. This (and the fact that $\mathcal{F}_0$ is trivial and thus $M(0) = 0$ a.s.) implies that the time of default is a totally inaccessible stopping time. Dropping this assumption (to allow discontinuities in A) one sees readily from the derivation of equation (2.38) that the defaultable spot rate $\overline{r}$ cannot be finite at jumps of A. One would have to specify the defaultable term structure in a more general way by defining a process $R^d(t) := \int_0^t \overline{r}(s)ds$ which will be well-defined and can account for the jumps in A. Similar definitions will be needed for the forward rates. Then (2.38) translates into $dR^d(t) = r(t)dt + dA(t)$. With this specification we can also drop the initial assumption that a default cannot be predicted with certainty.

It is important to note that the default-free term structure and the defaultable term structure must satisfy the conditions simultaneously. This will become clearer in the following version of theorem 2.4 that is set under the *statistical* measure P:

Theorem 2.5

If the dynamics are given under a subjective probability measure P the following are equivalent:

1.) *The dynamics are arbitrage-free.*
2.) *There are predictable processes $\theta_1(t), \ldots, \theta_n(t)$ and a strictly positive predictable process $\phi(t)$ that satisfy the regularity conditions of theorem 2.3 such that for all $t < T$:*
    (i) *The difference between default-free and defaultable short rate is ϕ times the hazard rate:*

$$\overline{r}(t) - r(t) = \phi(t)\lambda(t). \tag{2.47}$$

(ii) The defaultable and the default-free forward rates satisfy

$$-\overline{\alpha}(t,T) + \sum_{i=1}^{n} \overline{\sigma}_i(t,T) \int_t^T \overline{\sigma}_i(t,v)\,dv = \sum_{i=1}^{n} \overline{\sigma}_i(t,T)\theta_i(t) \tag{2.48}$$

$$-\alpha(t,T) + \sum_{i=1}^{n} \sigma_i(t,T) \int_t^T \sigma_i(t,v)\,dv = \sum_{i=1}^{n} \sigma_i(t,T)\theta_i(t) \tag{2.49}$$

a.s. for all $t < T$.

Proof: 1.) $\Leftrightarrow$ There is an equivalent martingale measure Q
$\Leftrightarrow$ (Theorem 2.3) There are predictable processes $\theta_1(t), \dots, \theta_n(t)$ and a strictly positive predictable process $\phi(t)$ that satisfy the regularity conditions of theorem 2.3 such that (using theorem 2.4):
(i) $\lambda^Q = \phi\lambda$ and $h^Q(t) = \overline{r}(t) - r(t) = \phi(t)\lambda(t)$.
(ii) $dW_i^Q = dW_i - \theta_i dt$ and

$$\begin{aligned} d\overline{f}(t,T) &= \sum_{i=1}^{n} \left(\overline{\sigma}_i(t,T) \int_t^T \overline{\sigma}_i(t,v)\,dv \right) dt + \sum_{i=1}^{n} \overline{\sigma}_i(t,T) dW_i^Q(t) \\ df(t,T) &= \sum_{i=1}^{n} \left(\sigma_i(t,T) \int_t^T \sigma_i(t,v)\,dv \right) dt + \sum_{i=1}^{n} \sigma_i(t,T) dW_i^Q(t). \end{aligned}$$

By substituting the P-dynamics of $\overline{f}(t,T)$ and $f(t,T)$ and equating coefficients the proof is concluded.

□

From theorem 2.5 one sees directly that there is only one set of market prices of risk for both the defaultable and the default-free term structure. This follows from the fact that there is only one set of underlying Brownian motions that drive the market. The market price of jump risk ϕ is uniquely determined by equation (2.47) which can be used as defining relationship for ϕ. Even with defaultable zero coupon bonds one can set up portfolios that are hedged against default risk (by making sure that the total value of the portfolio is zero), but still carry exposure to the Brownian motions. These portfolios must be related to the default-free term structure in their dynamics, and this relation is given in the two theorems above.

2.4 Modelling the Spread between the Forward Rates

When trying to connect the dynamics of the defaultable term structure and the default-free term-structure the most important relationship is (2.40):

$$0 \leq \lambda(t) = \overline{r}(t) - r(t).$$

In many cases a model of the default-free interest rates and forward rates will already be in place and the task is to find a specification of a model of the defaultable term structure that does not violate (2.40). If one directly estimated and implemented a model for the defaultable term structure $\overline{f}(t,T)$ without reference to the existing model of the default-free term structure, situations where $\overline{r} < r$ are bound to arise and the (combined) model will not be arbitrage-free.

A way around this problem is not to model the forward rates but the difference between these as proposed in Duffie (1994):

Definition 2.5
The continuously compounded instantaneous forward rate spread $h(t,T)$ *is defined as the difference between the defaultable forward rate and the default-free forward rate:*

$$h(t,T) = \overline{f}(t,T) - f(t,T). \tag{2.50}$$

Under the martingale measure we have $\lambda(t) = h(t,t)$.

Now one has to find a model for $h(t,T)$ which is compatible with theorem 2.4. The advantage of modelling $h(t,T)$ instead of $\overline{f}(t,T)$ is that (2.40) reduces to the well known problem of ensuring that $h(t,t) > 0$. and we can hope to use some of the extensive literature on interest rate models with positive short rates. We use the following dynamics for h:

Assumption 2.3
The dynamics of h are given by

$$h(t,T) - h(0,T) = \int_0^t \alpha^h(v,T)\,dv + \sum_{i=1}^n \int_0^t \sigma_i^h(v,T)dW^i(v). \tag{2.51}$$

Then

$$\overline{\alpha}(v,t) \quad = \quad \alpha(v,t) + \alpha^h(v,t) \tag{2.52}$$

$$\overline{\sigma}_i(v,t) \;=\; \sigma_i(v,t) + \sigma_i^h(v,t). \tag{2.53}$$

In place of the drift restriction (2.42) we reach:

Corollary 2.6
Let the default-free interest rates satisfy the HJM drift restriction (2.43). A model for the defaultable forward rates based on the forward rate spread $h(t,T)$ must imply under the martingale measure

$$\begin{aligned}\alpha^h(t,T) \;=\; \sum_{i=1}^{n}\Bigg[&\sigma_i(t,T)\int_t^T \sigma_i^h(t,v)\,dv \\ &+\sigma_i^h(t,T)\int_t^T \sigma_i(t,v)\,dv \\ &+\sigma_i^h(t,T)\int_t^T \sigma_i^h(t,v)\,dv\Bigg].\end{aligned} \tag{2.54}$$

Proof: Substitute (2.52) and (2.53) in (2.42).

□

Again – as in the original HJM model – the drift of the spread is given in terms of the volatilities of the interest rates and spreads. Given these drift specifications one has to require that the process $h(t,t)$ is nonnegative. This will enable us to add a defaultable interest rate model to a given model of the default-free interest rate in a modular fashion.

If one chooses a specification of the dynamics of h that has nonnegative $h(t,t)$ a.s. under the subjective measure, this will ensure that $h(t,t)$ will be nonnegative a.s. under the martingale measure, too.

It will be interesting to analyse some possible specifications and the problems that may arise when modeling the spread structure.

2.4.1 Independence of Spreads and default-free Rates

The easiest way to specify the spreads is to avoid the cross-variation terms with the default-free term-structure in (2.54). Assume that every factor W_i either influences f or h but never both. Then $\forall\, i = 1,\ldots,n$

$$\sigma_i^h(t,T) \neq 0 \qquad \Rightarrow \qquad \int_t^T \sigma_i(t,v)\,dv = 0$$

$$\sigma_i(t,T) \neq 0 \qquad \Rightarrow \qquad \int_t^T \sigma_i^h(t,v)\,dv = 0 \tag{2.55}$$

and the drift restriction for the spreads becomes the usual HJM restriction:

Corollary 2.7

(i) If σ_i^h and σ_i satisfy (2.55) then (under the martingale measure)

$$\alpha^h(t,T) = \sum_{i=1}^{n} \sigma_i^h(t,T) \int_t^T \sigma_i^h(t,v)dv \tag{2.56}$$

and $h(t,t) > 0$ a.s. are necessary and sufficient for absence of arbitrage.

(ii) Equation (2.55) is satisfied if $h(t,T_1)$ and $\overline{f}(t,T_2)$ are independent for all $t \leq T_1,\ t \leq T_2$, i.e. the term structure of the spreads and the term structure of the default-free forward rates are independent.

Proof: The first part follows directly by substituting the assumptions (2.55) in (2.54). For the last part observe that independence of the term structures of spreads and default-free rates implies that

$$\sigma_i(t,T_1)\sigma_i^h(t,T_2) = 0$$

for (almost) all $t \leq T_1, T_2$ which in turn implies (2.55) directly.

□

Note that strict independence of h and g is not needed. One might imagine a model where the term structure of the spreads is driven by an additional Brownian motion alone (this will ensure (2.55)), but the volatility of the spread might still depend on the level of the default-free interest rates.

Satisfying the positivity requirement (2.40) on $h(t,t)$ becomes very easy in the setup of corollary 2.7: One can use any interest rate model for $h(t,T)$ that is known to generate positive short rates, e.g. the square root model of Cox, Ingersoll and Ross (1985b) or the model with lognormal interest rates by Sandmann and Sondermann (1997).

2.4.2 Negative Forward Spreads

There is one additional caveat when using the nonnegative rate model for the forward rate spreads: Even though we require that the 'short' spread $h(t,t) > 0$ is greater than zero, a 'forward' spread $h(t, T)$ $(T > t)$ might still become negative.

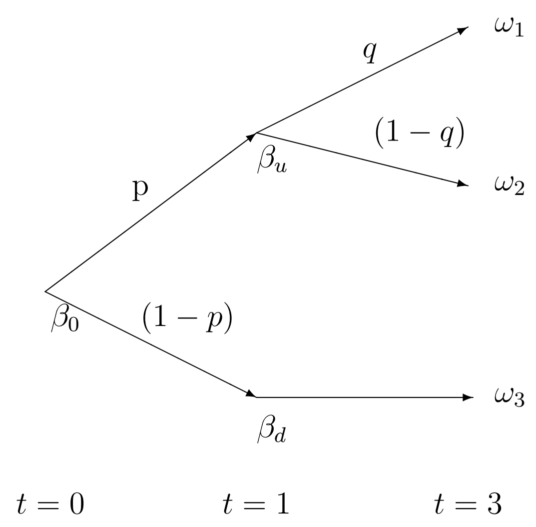

Figure 2.1: Example for negative forward spreads

As an example how this can arise we consider the two-period economy with points in time $t = 0, 1, 2$ from figure 1. There are three states $\omega_1, \omega_2, \omega_3$ and the filtration is $\mathcal{F}_0 = \{\{\omega_1, \omega_2, \omega_3\}, \emptyset\}$; $\mathcal{F}_1 = \sigma(\{\omega_1, \omega_2\}, \omega_3)$; $\mathcal{F}_2 = \sigma(\omega_1, \omega_2, \omega_3)$. The states have risk-neutral probabilities $P(\omega_1) = pq$; $P(\omega_2) = p(1-q)$; $P(\omega_3) = (1-p)$. There are default-free bonds $B(0,1), B(0,2)$ and defaultable bonds $\overline{B}(0,1), \overline{B}(0,2)$. In state ω_1 both defaultable bonds survive, in ω_2 only the defaultable bond with maturity 2 defaults and in ω_3 both bonds default (with zero recovery).

The initial default-free term structure is given by $B(0,1) = \beta_0$ and $B(0,2) = \beta_0(p\beta_u + (1-p)\beta_d)$. Thus the bond prices are

$$\begin{aligned}
B(0,1) &= \beta_0 \\
B(0,2) &= \beta_0(p\beta_u + (1-p)\beta_d) \\
\overline{B}(0,1) &= p\,\beta_0 \\
\overline{B}(0,2) &= p\,q\,\beta_0\beta_u
\end{aligned}$$

The forward rates are

$$\begin{aligned} f(0,1,2) &= -\ln(p\,\beta_u + (1-p)\,\beta_d) \\ \overline{f}(0,1,2) &= -\ln(q\,\beta_u). \end{aligned}$$

We have a negative forward spread $h(0,1,2) < 0$ or $\overline{f}(0,1,2) < f(0,1,2)$ if

$$p\,\beta_u + (1-p)\,\beta_d < q\,\beta_u \tag{2.57}$$

holds. This is equivalent to

$$q > p + (1-p)\frac{\beta_d}{\beta_u}, \tag{2.58}$$

so a necessary condition for negative forward spreads is that $\beta_d < \beta_u$. Choose for instance $p = 0.9, \quad \beta_d/\beta_u = 0.9, \quad q = 0.995.$

This example can be regarded as a 'snapshot' from a continuous-time model in which the relevant prices and probabilities have been aggregated to the two-period example.

The condition $\beta_d < \beta_u$ means that $r_d > r_u$. For negative forward spreads to arise we need
– either q and p are of the same order of magnitude, then the ratio β_d/β_u must be very small, $r_u << r_d$,
– or q is much larger than p, then β_u can be of the same order as β_d. In practice $q >> p$ only occurs if $T_1 >> T_2 - T_1$. But then $\beta_d/\beta_u \sim 1$ because of the short horizon $T_2 - T_1$ which is very far in the future, as well.

The occurrence of negative forward spreads is due to the special way in which we defined the defaultable forward rates. It is not possible to exploit the negative forward spread as an arbitrage-opportunity because the portfolio one would typically use for that will be destroyed by an early default:

In the default-free bonds one can set up a portfolio that replicates the payoff of a default free forward contract, but set up in defaultable bonds this portfolio disappears in the case of an early default. If one had gone long a default-free forward contract and short a *replicating portfolio* of a *defaultable* forward contract, an early default (which eliminates the replicating portfolio for the defaultable forward contract) leaves one with the default-risk free half of the portfolio, which now is exposed to changes in the default-free term structure. If the subsequent default-free interest rates are high the remainder of the portfolio will generate a loss.

Summarizing, negative forward spreads can only occur if there is a strong correlation between early default (event ω_3) and high interest rates (β_d small), and a strong correlation between early survivial (events $\{\omega_1, \omega_2\}$) and low interest rates (β_u large).

2.5 Positive Recovery and Restructuring

In the preceding sections we assumed that a defaultable zero coupon bond has a payoff of zero upon default. This assumption is unnecessarily restrictive and does not agree with market experience.

As mentioned in the introduction, real-world defaults often have the following features.
Positive recovery: Franks and Torous (1994) find recovery rates between 40 and 80 percent.
Reorganisation: The majority of firms in financial distress are reorganised and re-floated, they are not liquidated.
Compensation in terms of new securities: On average about one third of the compensation to the holders of defaulted bonds is in cash , two thirds are in terms of new securities of the defaulted and restructured firm.
Multiple defaults: A reorganised firm can default again. We have the possibility of multiple defaults with in principle any number of defaults (each with subsequent restructuring of the defaulted firm's debt).

The main results of the preceding sections are still valid if the recovery of the bond is positive and not zero. We choose the following setup including the possibility of multiple defaults:

If a default occurs a *restructuring* of the debts occurs. Holders of the old debt loose a fraction of q of their claims, where $q \in [0, 1]$ is possibly unpredictable, but known at default.

> A pre-default claim of \$ 1 face value becomes a claim of \$ $(1 - q)$ face value after the default. The maturity of the claim remains unchanged.

This model mimicks the effect of a rescue plan as it is described in many bankruptcy codes: The old claimants have to give up some of their claims in order to allow for rescue capital to be invested in the defaulted firm. They are *not* paid out in cash [8] (this would drain the defaulted firm of valuable liquidity) but in 'new' defaultable bonds of the same maturity.

[8]The holder of a defaulted bond is free to sell this bond on the market, though.

As the vast majority of defaulted debtors continue to operate after default, a representation of the loss of a defaulted bond in terms of a reduction in face value is possible even if the actual payoff procedure is different.

For ease of modelling we use the convention that the *defaultable forward rates are quoted with respect to a bond of face value 1 $.*

The reduction in the face value of a bond in default is *not* reflected in the forward rates. This convention enables us to separate the effects of changes in interest rates (representing expectations on future defaults) and the direct effect of the default.

This modelling approach has much in common with the *fractional recovery* introduced by Duffie and Singleton (1997; 1999) and Duffie et.al. (1994; 1996; 1994).

Duffie specifies the payoff in default to be a predictable fraction $(1-q)$ of the value of a 'non-defaulted but otherwise equivalent security'. This is inspired by the default procedures in swap contracts. In Duffie's mathematical model the value V_τ of the defaulted security directly after default (i.e. the payoff in default) is specified as $V_\tau := (1-q)V_{\tau-}$, the fraction $(1-q)$ times the value of the same security directly before default. [9]

Furthermore we include *magnitude risk* in our setup. The magnitude of the default is uncertain and the actual realisation of the loss q_i need not be predictable, it can be considered as a random draw at τ_i from the distribution $K(dq)$. This distribution may itself be stochastic.

The analysis of the pricing of defaultable zero coupon bonds goes along the lines of the zero-recovery case, where we increasingly have to introduce the theory of marked point processes. Standard references are Jacod and Shiryaev (1988), and Bremaud (1981) for the mathematical theory and Jarrow and Madan (1995) and Björk, Kabanov and Runggaldier (1996) and Björk, Di Masi, Kabanov and Runggaldier (1997) for the application to interest rate theory[10].

2.5.1 The Model Setup

Mathematically the setup is as follows:

[9] The model presented here can be extended to include the possibility of predictable times of default. (See the remarks at the end of section 5.) In addition to this we avoid the backward recursive stochastic integral equations that are necessary in the Duffie model.

[10] For other financial applications see also Merton (1976).

Assumption 2.4

(i) Defaults occur at the stopping times $\tau_1 < \tau_2 < \ldots$.

(ii) At each time τ_i *of default a loss quota* $q_i \in E$ *is drawn from a measurable space* $(E, \mathcal{E})$, $E \subset I\!R$, *the* mark space. *(Usually* $E = [0,1]$ *with the Borel sets.)*

(iii) The double sequence $(\tau_i, q_i), i \in I\!N_+$ *defines a marked point process* [11] *with defining measure*

$$\mu(\omega, t;\, dq, dt) \tag{2.59}$$

and predictable compensator

$$\nu(\omega, t)(dq, dt) = K(\omega, t)(dq)\ \lambda(t)\, dt \tag{2.60}$$

(iv) Consider the defaultable zero coupon bond $\overline{B}(0, T)$. *At time* T, *the maturity of this bond, it pays out*

$$Q(T) := \prod_{\tau_i \le T} (1 - q_i), \tag{2.61}$$

the remainders after all fractional default losses. [12] $Q(t)$ *can be represented as a Doleans-Dade exponential:* $Q(0) = 1$ *and*

$$\frac{dQ(t)}{Q(t-)} = -\int_0^1 q\, \mu(dq, dt). \tag{2.62}$$

(v) The filtration $(\mathcal{F}_t)_{(t \ge 0)}$ *is generated by the Brownian motions* W^i *and the marked point process* μ.

(vi) We assume sufficient regularity on the marked point process μ *to justify all subsequent manipulations.*

– The sequence of default times is nonexplosive.

– μ *is a multivariate point process (see (1988)).*

– $\int_0^t \int_E K(dq)\lambda(s)\, ds < \infty$ *for all* $t < \infty$.

– The processes introduced in definition 2.6 are square-integrable.

– The resulting bond prices are bounded.

For the subsequent analysis we need to define the following processes:

Definition 2.6

The loss summation function $N'(t)$, *the instantaneous expected loss rate* $q(t)$, *the default*

[11] For a general reference on marked point processes see Jacod and Shiryaev (1988) and Bremaud (1981).

[12] It will be clear from the context whether Q denotes the martingale measure or the accumulated fractional default losses. The latter will usually be the case from now on.

compensator $A'(t)$ and the default martingale $M'(t)$ are defined as:

$$N'(t) := \int_0^t \int_0^1 q\,\mu(dq, ds) \tag{2.63}$$

$$q(t) := \int_0^1 q\,K(dq) \tag{2.64}$$

$$A'(t) := \int_0^t \int_0^1 q\,K(dq)\lambda(s)ds = \int_0^t q(s)\lambda(s)\,ds \tag{2.65}$$

$$M'(t) := N'(t) - A'(t). \tag{2.66}$$

Note that A' is the predictable compensator of N', and M' is a martingale, and $dQ(t)/Q(t-) = -dN'(t)$.

Assumption 2.5
In the presence of multiple defaults:

(i) The dynamics of the defaultable rates are given by assumption 2.1.

(ii) The dynamics of the defaultable bond prices are as in assumption 2.1, with N' replacing N:

$$\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = \overline{\mu}(t,T)dt + \sum_{i=1}^{n} \overline{\eta}_i(t,T)dW^i(t) - dN'.$$

(iii) The defaultable bank account is $c(t) = Q(t)\exp\{\int_0^t \overline{r}(s)\,ds\}$.

(iv) The dynamics of the default risk free rates and bond prices are given by assumption 2.2.

(v) There is no total loss: $q_i < 1$ a.s.

Assumption 2.5 implies that all forward rates and the process $G(t,T) = \int_t^T \overline{f}(t,s)\,ds$ are continuous at times of default.

For the default-free rates this is justifiable in most cases (unless a large sovereign debtor is concerned), but the defaultable rates should be modelled by explicitly allowing for dependence on the defaults μ. A default will usually discontinuously change the market's estimation of the future likelihood of defaults and thus the defaultable forward rates. This effect will be included in a later section, here we assume that the only direct effect of a default is the reduction of the face value of the defaultable debt. Nevertheless we allow the default to influence the diffusion parameters of the forward rates which distinguishes this setup from the literature on Cox processes (see Lando (1998)).

2.5.2 Change of Measure

First, Girsanov's theorem (theorem 2.3) takes the following form: [13]

Theorem 2.8
Let θ be a n-dimensional predictable processes $\theta_1(t), \dots, \theta_n(t)$ and $\Phi(t,q)$ a strictly positive predictable function [14] with

$$\int_0^t \|\theta(s)\|^2 ds < \infty, \qquad \int_0^t \int_E |\Phi(s,q)|\, K(dq)\, \lambda(s) ds < \infty$$

for finite t. Define the process L by $L(0) = 1$ and

$$\frac{dL(t)}{L(t-)} = \sum_{i=1}^{n} \theta_i(t) dW^i(t) + \int_E (\Phi(t,q) - 1)(\mu(dq,dt) - \nu(dq,dt)).$$

Assume that $\mathbf{E}\,[\, L(t)\,] = 1$ for finite t.
Then there is a probability measure Q equivalent to P with

$$dQ_t = L_t dP_t \tag{2.67}$$

such that

$$dW(t) - \theta(t)dt = d\widetilde{W}(t) \tag{2.68}$$

defines $\widetilde{W}$ as Q-Brownian motion and

$$\nu_Q(dq,dt) = \Phi(t,q)\nu(dq,dt) \tag{2.69}$$

is the predictable compensator of μ under Q.
Every probability measure that is equivalent to P can be represented in the way given above.

Proof: See BKR (1996).

□

The only change to theorem 2.3 is the new predictable compensator of the marked point process which has the form $\Phi(t,q)\nu(dq,dt)$. Instead of a single market price of risk for the jump risk we now have a market price of risk for each subset $e \in \mathcal{E}$ of the marker space.

[13] See Jacod and Shiryaev (1988) and Björk, Kabanov and Runggaldier (1996).
[14] In functions of the marker q (like Φ here) *predictability* means measurable with respect to the σ-algebra $\tilde{\mathcal{P}} := \mathcal{P} \otimes \mathcal{E}$. Here $\mathcal{P}$ is the σ-algebra of the predictable processes. See Jacod and Shiryaev (1988) for details.

The market price of risk of a default with loss $q \in e$ is then $\int_e \Phi(q,t)K(dq)/\int_e K(dq)$ per unit of probability.

Note that now we have a much larger class of potential martingale measures, as for every (t,ω) a *function* $\Phi(t,q)$ has to be chosen and not just the value of the process $\phi(t)$. Typically we will have incomplete markets in this situation which poses entirely new problems for the hedging of contingent claims. See Björk, Kabanov and Runggaldier (1996) for a detailed analysis of trading strategies, hedging and completeness in bond markets with marked point processes.

As before, to save notation, we will assume that all dynamics are already specified with respect to the martingale measure.

2.5.3 Dynamics and Absence of Arbitrage

We start from the representation of the defaultable bond prices as (using the notation and results of the preceding sections)

$$\overline{B}(t,T) = \exp\left\{-G(t,T)\right\} Q(t).$$

The dynamics of $\overline{B}(t,T)$ are then

$$\begin{aligned}
\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} &= -dG(t,T) + \frac{1}{2}d < G, G > -dN' \\
&= \left[-\overline{\gamma}(t,T) + \overline{r}(t) + \frac{1}{2}\sum_{i=1}^{n}\overline{a}_i^2(t,T)\right] dt \\
&\quad + \sum_{i=1}^{n}\overline{a}_i(t,T)dW^i(t) - dN'(t) \\
&= \left[-\overline{\gamma}(t,T) + \overline{r}(t) - q(t)\lambda(t) + \frac{1}{2}\sum_{i=1}^{n}\overline{a}_i^2(t,T)\right] dt \\
&\quad + \sum_{i=1}^{n}\overline{a}_i(t,T)dW^i(t) - dM'(t),
\end{aligned}$$

where we used that G is continuous. Absence of arbitrage is here equivalent to

$$r(t) = -\overline{\gamma}(t,T) + \overline{r}(t) - q(t)\lambda(t) + \frac{1}{2}\sum_{i=1}^{n}\overline{a}_i^2(t,T). \tag{2.70}$$

To show that the results of the preceding sections remain valid we only need to show that

$$\lambda(t)q(t) = \overline{r}(t) - r(t). \tag{2.71}$$

The argument goes exactly as before: The Doleans-Dade exponential of the martingale $M'(t) = N'(t) - \int_0^t q(s)\lambda(s)\, ds$ is

$$\exp\{-\int_0^t q(s)\lambda(s)ds\} \prod_{T_i \le t} (1 - q_i), \tag{2.72}$$

while the discounted value of the defaultable bank account is the Q-martingale

$$\overline{c}(t) := \frac{c(t)}{b(t)} = \exp\{\int_0^t \overline{r}(s) - r(s)\, ds\} \prod_{T_i \le t} (1 - q_i). \tag{2.73}$$

This is the Doleans-Dade exponential of

$$\hat{M}(t) := -N'(t) + \int_0^t \overline{r}(s) - r(s)\, ds. \tag{2.74}$$

Because $q_i < 1$ a.s. we have that $\overline{c}(t) > 0$ a.s. and therefore $\hat{M}$ is unique and well-defined as $\hat{M}(t) = \int_0^t \frac{d\overline{c}(s)}{\overline{c}(s-)}$. Again, we see that

$$M'(t) + \hat{M}(t) = \int_0^t \overline{r}(s) - r(s) - q(s)\lambda(s)\, ds \equiv 0 \tag{2.75}$$

(being a predictable martingale with initial value zero) must be constant and equal to zero. Therefore

$$\lambda(t)q(t) = \overline{r}(t) - r(t) > 0. \tag{2.76}$$

Equation (2.76) is the equivalent of equation (2.38), the key relationship which allowed for the derivation of conditions for the absence of arbitrage. These conditions are exactly the same as for zero recovery, the proof is the same as for theorem 2.4.

Theorem 2.9
The following are equivalent:

1. *The measure under which the dynamics are specified is a martingale measure.*
2. *(i) The short interest rate spread is the intensity of the default process multiplied*

with the locally expected loss quota. It is positive (for $q(t) > 0$).

$$q(t)\lambda(t) = \overline{r}(t) - r(t) > 0. \tag{2.77}$$

(ii) The drift coefficients of the defaultable forward rates satisfy for all $t \leq T$ equations (2.41) and (2.42).

(iii) The drift coefficients of the default-free forward rates satisfy for all $t \leq T$ equation (2.43).

3. (i) $\overline{r}(t) - r(t) = \lambda(t)q(t) > 0$.

(ii) The dynamics of the defaultable bond prices are given by

$$\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} = \overline{r}(t)dt + \sum_{i=1}^{n} \overline{a}_i(t,T)dW^i(t) - dN'(t) \tag{2.78}$$

or, solving the s.d.e.

$$\begin{aligned} \overline{B}(t,T) &= \overline{B}(0,T)\,Q(t)\,\exp\left\{\int_0^t \overline{r}(s)\,ds - \frac{1}{2}\sum_{i=1}^{n}\int_0^t \overline{a}_i^2(s,T)\,ds \right. \\ &\left. + \sum_{i=1}^{n}\int_0^t \overline{a}_i(s,T)\,dW^i(s)\right\}. \end{aligned} \tag{2.79}$$

(iii) The dynamics of the default-free bonds satisfy (2.46).

All no-arbitrage restrictions on the dynamics of the interest rates are exactly identical to the restrictions in theorem 2.4, although theorem 2.4 only concerned the situation with zero recovery. Thus theorem 2.9 allows us to directly transfer all results on the modelling of the spreads. The drift restrictions of the corollaries 2.6 and 2.7 and of theorem 2.5 are also valid in the present setup.

For the modelling of arbitrage-free dynamics of the defaultable interest rates $\overline{f}(t,T)$ one need not be concerned with the specification of the recovery rates, it is sufficient to just model the interest rates subject to the positive spread restriction (2.40) or (2.77) and the drift restrictions (2.42) and (2.43). Again we see that the hard task of modelling an unobservable quantity (like the distribution of the loss quota q) can be replaced with a suitable model of the defaultable forward rates which are much more easily observed.

If one allows q to take on negative values, negative spot spreads $\overline{r} - r$ are possible. A negative q means that the defaultable bond gains in value upon default. Of course such an event is very rare but in some cases there might be an early (and full) repayment of the debt which will result in $q < 0$, for instance if the proceeds of a liquidation are greater than the outstanding debt or if the default event is caused by a strategic default or a takeover. The advantage of negative q is to allow a wider class of models to be used for $q\lambda$, e.g. particularly the Gaussian models like the models of of Vasicek (1977) and Ho and Lee (1986).

2.5.4 Seniority

Bonds of different seniority have different payoffs in default, the ones with higher seniority have a higher payoff than the ones with lower seniority. Strict seniority – junior debt has a positive payoff if and only if senior debt has full payoff – is rather rare in practice which is due to the various legal bankruptcy procedures, but in general senior debt has a higher payoff in default than junior debt.

With a loss quota of junior debt q^j that is higher than the loss of senior debt q^s, the defaultable instantaneous short rate of junior debt is greater than the short rate for senior debt: $\overline{r}^j > \overline{r}^s$.

For modelling junior and senior debt another stage is added to the usual defaultable debt modelling. First model the default-free term-structure. Then model the spread to the senior bonds. Then (this is the new step) model the spread between junior and senior debt using the senior debt as 'default-free' debt in the drift restrictions. The modelling restrictions we derived above still hold in this setup.

2.6 Instantaneous Short Rate Modelling

Going back to the representation (2.79) of the dynamics of the defaultable bond prices

$$\begin{aligned}\overline{B}(t,T) &= \overline{B}(0,T) \cdot \prod_{\tau_i \le t}(1-q_i) \exp\left\{\int_0^t \overline{r}(s)\, ds\right\} \\ &\cdot \quad \exp\left\{-\frac{1}{2}\sum_{i=1}^n \int_0^t \overline{a}_i^2(s,T)\, ds + \sum_{i=1}^n \int_0^t \overline{a}_i(s,T)\, dW^i(s)\right\},\end{aligned}$$

one can evaluate this expression at $t = T$ and use the final condition $\overline{B}(T,T) = \prod_{\tau_i \le T}(1 - q_i)$ to reach

$$\begin{aligned} & \prod_{\tau_i \le T} (1-q_i) \exp\left\{ - \int_0^T \overline{r}(s)\, ds \right\} \\ = \; & \overline{B}(0,T) \exp\left\{ -\frac{1}{2} \sum_{i=1}^n \int_0^T \overline{a}_i^2(s,T)\, ds + \sum_{i=1}^n \int_0^T \overline{a}_i(s,T)\, dW^i(s) \right\} \\ \cdot \; & \prod_{\tau_i \le T} (1-q_i). \end{aligned} \tag{2.80}$$

If $q < 1$ a.s. we may divide both sides by $\prod_{\tau_i \le T}(1 - q_i)$ and take expectations of both sides to reach

Corollary 2.10

If there is no total loss on the defaultable bond (i.e. $q_i < 1$), we have the following representation of the price of defaultable zero coupon bonds:

$$\overline{B}(0,T) = \mathbf{E}\left[\exp\left\{ - \int_0^T \overline{r}(s)\, ds \right\} \mid \mathcal{F}_0 \right]. \tag{2.81}$$

Here we used that the second exponential is a stochastic exponential of the martingale $\sum_i \int \overline{a}_i(s,T) dW^i(s)$. Thus it is again a martingale with initial value 1.

In the first sections with zero recovery we were not able to derive this representation as $\overline{r}$ was not defined for times after the default. The above representation of the prices of defaultable bonds is the exact analogue to the representation of the prices of default-free bonds $B(t,T) = \mathbf{E}\left[\exp\{-\int_t^T r(s)ds\} \right]$ as discounted expected value of the final payoff 1. This representation is the starting point of all models of the term-structure of interest rates that are based on a model of the short rate[15]. A result of this type has been proved by Duffie, Schroder and Skiadas (1994) (but in their valuation formula an additional jump term occurs) and by Lando (1998) for the case of a default that is triggered by the first jump of a Cox process. Here the Cox process assumption is not needed, the default process can have an intensity that conditions on previous defaults.

Alternatively to the modeling of defaultable interest rates in the HJM- framework of assumption 2.1 one can model the short rates directly. With any arbitrage-free short rate

[15]Popular models of the short rate are by (among others): Vasicek (1977), Cox, Ingersoll and Ross (1985b), Ho and Lee (1986), Black, Derman, Toy (1990), Hull and White (1993) and Sandmann and Sondermann (1997).

model for the default-free short rate r and a positive short rate model for the spread h one can specify an arbitrage-free model framework. Because the model for the defaultable short rate will necessarily be at least a two-factor model, the calibration of this model might become difficult and the HJM approach may be preferable. On the other hand the short rate models need not worry about possibly negative forward spreads and are better suited for analysis with partial differential equations.

2.7 Jumps in the Defaultable Rates

In the presence of multiple defaults (with ensuing restructuring) it is more realistic to allow the defaultable rates to change discontinuously at times of default, the defaultable term structure must be allowed to change its *shape* at these events.

These jumps in the defaultable rates are not to be confused with the fractional loss at default. There are two distinct effects at a time of default which both cause a discrete change in the value process of the holders of defaultable bonds:
First, there is the direct loss caused by the rescue plan and the reduction of the claims of the defaulted bondholders. This is modeled by the marker process q.
Second, the market's valuation of the defaultable bonds may change due to the default. This is reflected in a discrete change in the yield curve but need not mean a irrecoverable loss. If there is no further default until maturity this jump in the value process is compensated[16].

2.7.1 Dynamics

To reach the most general setup we use the marked point process $\mu(dq, dt)$. At every default there are jumps in the defaultable term structure $\overline{f}(t,T)$ and the defaultable bond prices $\overline{B}(t,T)$.

We replace assumption 2.1 with:

[16]The general methodology of modelling interest rates in the presence of marked point processes is taken from Björk, Kabanov and Runggaldier (1996), who give an excellent account on default-free interest rate modelling with marked point processes.

Assumption 2.6
(i) The dynamics of the defaultable forward rates are:

$$d\overline{f}(t,T) = \overline{\alpha}(t,T)dt + \sum_{i=1}^{n} \overline{\sigma}_i(t,T)\, dW^i(t) + \int_E \delta(q;t,T)\,\mu(dq,dt) \tag{2.82}$$

(ii) The dynamics of $\widetilde{B}(t,T) := \exp\{-\int_t^T \overline{f}(t,s)ds\}$ *are:*

$$\begin{aligned} \frac{d\widetilde{B}(t,T)}{\widetilde{B}(t-,T)} &= m(t,T)\,dt + \sum_{i=1}^{n} \overline{a}_i(t,T)\,dW^i(t) \\ &\quad + \int_E \theta(q;t,T)\,\mu(dq,dt) \end{aligned} \tag{2.83}$$

(iii) The dynamics of the defaultable bond prices $\overline{B}(t,T) = Q(t)\widetilde{B}(t,T)$ *are:*

$$\begin{aligned} \frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} &= m(t,T)\,dt + \sum_{i=1}^{n} \overline{a}_i(t,T)\,dW^i(t) \\ &\quad + \int_E (1-q)\theta(q;t,T)\,\mu(dq,dt) - \int_E q\,\mu(dq,dt) \end{aligned} \tag{2.84}$$

(iv) We assume sufficient regularity on the parameters to allow:
– differentiation (w.r.t. T) *under the integral,*
– interchange of order of integration,
– finite prices $\overline{B}(t,T)$ *almost surely.*

We distinguish between a 'pseudo' bond price $\widetilde{B}(t,T)$ in which the influence of previous defaults has been removed, and the 'real' bond price $\overline{B}(t,T) = Q(t)\widetilde{B}(t,T)$. The dynamics of $\overline{B}(t,T)$ in (2.84) follow directly from Itô's lemma.

These dynamics are interdependent due to the following result by BKR (1996):

Proposition 2.11
Given the dynamics (2.82) of $\overline{f}(t,T)$
(i) the dynamics of the defaultable short rate $\overline{r}(t)$ *are*

$$\begin{aligned} d\overline{r}(t) &= [\frac{\partial}{\partial T}\Big|_{T=t} \overline{f}(t,T) + \overline{\alpha}(t,t)]dt + \sum_{i=1}^{n} \overline{\sigma}_i(t,t)dW^i(t) \\ &\quad + \int_E \delta(q;t,t)\mu(dq,dt). \end{aligned} \tag{2.85}$$

(ii) the dynamics of $\widetilde{B}(t,T)$ are

$$m(t,T) = \overline{r}(t) - \int_t^T \overline{\alpha}(s,T)\,ds + \frac{1}{2}\sum_{i=1}^{n}(\int_t^T \overline{\sigma}_i(s,T)\,ds)^2 \quad (2.86)$$

$$\overline{a}_i(t,T) = -\int_t^T \overline{\sigma}_i(s,T)\,ds \quad (2.87)$$

$$\theta(q;t,T) = \exp\left\{-\int_t^T \delta(q;s,T)\,ds\right\} - 1. \quad (2.88)$$

(iii) The dynamics of the defaultable bond prices are given by assumption 2.6 (iii) with the specification of (ii) above.

Proof: See BKR (1996) for (i) and (ii), point (iii) follows directly.

□

2.7.2 Absence of Arbitrage

The change of measure to the martingale measure is done according to theorem 2.8. The analysis leading to the key relation (2.77)

$$\lambda(t)q(t) = \overline{r}(t) - r(t) > 0$$

in section 2.5.3 is still valid, because the only defaultable security needed there is the defaultable bank account $c(t)$ which has no jump component in its development except the direct losses of q_i at default.

As usual we need for absence of arbitrage

$$\begin{aligned} r(t)\,dt &= \mathbf{E}\left[\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)}\right] \\ &= \overline{r}(t)dt - \overline{\gamma}(t,T)\,dt + \frac{1}{2}\sum_{i=1}^{n}\overline{a}_i^2(t,T)\,dt \\ &\quad + \int_E \theta(q;t,T)(1-q)\,K(dq)\lambda(t)\,dt - q(t)\lambda(t)dt. \end{aligned}$$

Substituting (2.77) and (2.88) yields:

Proposition 2.12
Under the martingale measure

(i) The short rate spread is given by

$$\overline{r}(t) - r(t) = \lambda(t)q(t). \tag{2.89}$$

(ii) The drift of the defaultable forward rates is restricted by

$$\overline{\gamma}(t,T) = \frac{1}{2}\sum_{i=1}^{n}\overline{a}_i^2(t,T) + \int_E \left(\exp\left\{-\int_t^T \delta(q;t,v)\,dv\right\} - 1\right)(1-q)\,K(dq)\lambda(t), \tag{2.90}$$

or, differentiated,

$$\begin{aligned}\overline{\alpha}(t,T) &= \sum_{i=1}^{n}\overline{\sigma}_i(t,T)\int_t^T \overline{\sigma}_i(t,v)\,dv \\ &- \int_E \delta(q;t,T)\exp\left\{-\int_t^T \delta(q;t,v)\,dv\right\}(1-q)\,K(dq)\lambda(t).\end{aligned} \tag{2.91}$$

(iii) The dynamics of the defaultable bond prices under the martingale measure are

$$\begin{aligned}\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)} &= \overline{r}(t,T)\,dt + \sum_{i=1}^{n}\overline{a}_i(t,T)\,dW^i(t) \\ &+ \int_E (1-q)\theta(q;t,T)\left(\mu(dq,dt) - K(dq)\,\lambda(t)\,dt\right) \\ &-dN'(t)\end{aligned} \tag{2.92}$$

where $\theta(q;t,T)$ is defined as in (2.88).

Obviously the drift restriction (2.90) cannot be handled as easily as the other restrictions in theorems 2.4 and 2.9 before because of the integral over the jumps of the forward rates. As the defaults now have a jump-influence on the defaultable forward rates, the parameters of the default process do not disappear any more.

BKR (1996) and Jarrow and Madan (1995) reach a quite similar restriction to (2.91) for the modelling of default-free interest rates in the presence of marked point processes. The restriction of BKR is

$$\begin{aligned}\alpha(t,T) &= \sum_{i=1}^{n}\sigma_i(t,T)\int_t^T \sigma_i(t,v)\,dv \\ &- \int_E \delta'(q;t,T)\exp\left\{-\int_t^T \delta'(q;t,v)\,dv\right\}K(dq)\lambda(t),\end{aligned}$$

and applies to the default-free interest rates. In this setup we assumed that the default-free interest rates do not jump (i.e. $\delta' = 0$) which reduces the restriction to the usual HJM-restriction (2.43).

The s.d.e. of the defaultable bond prices is of the usual type: There is a drift component of $\overline{r}$ and the default influence $-dN'(t)$. The other parts of the dynamics of the defaultable bond prices are local martingales.

2.8 Ratings Transitions

In the previous section we only allowed jumps in the term structure of defaultable bond prices at defaults. But in an environment where changes in credit ratings can cause large, sudden changes in the prices of defaultable bonds, one would like to be able to incorporate jumps in the term structure of defaultable bond prices that are not caused by defaults but only by rating transitions. This aim is easy to accomplish in the framework that has been laid in the previous sections.

First we have to define defaultable bond prices and forward rates for all rating classes:

Definition 2.7
There are K different rating classes for defaultable bonds. ('Default' is not counted as a rating class.) For each rating class $k = 1, \ldots, K$ there is a term structure of defaultable (pseudo) bond prices $\widetilde{B}_k(t,T)$ and defaultable forward rates $\overline{f}_k(t,T)$

$$\widetilde{B}_k(t,T) = \exp\{-\int_t^T \overline{f}_k(t,s)ds\}. \tag{2.93}$$

The defaultable (pseudo) bond prices $\widetilde{B}_k(t,T)$ and forward rates $\overline{f}_k(t,T)$ for all classes can be observed at all times.
To every defaultable bond $\overline{B}(t,T)$ there is a ratings process $R(t)$ which gives the rating of the bond at time t. Thus its price is given by

$$\overline{B}(t,T) = Q(t)\widetilde{B}_{R(t)}(t,T).$$

Note that there is a difference between the price of *one given* defaultable bond $\overline{B}(t,T)$ and the respective 'pseudo' bond price $\widetilde{B}_k(t,T)$ of the ratings class $k = R(t)$ in which the defaultable bond happens to be at this time. First, the pseudo-bond price never changes its rating (it is only used to define the bond prices in this rating class), and second, the pseudo bond price is not affected by defaults.

Next we describe the rating transition dynamics of the defaultable bond prices and the dynamics of the pseudo defaultable bond prices for all rating classes. Note that in contrast to most of the other literature we do not include 'default' as $K+1$st rating class, but prefer to model defaults using the multiple default model of the previous section. Assumption 2.6 is replaced with:

Assumption 2.7

(i) The rating transitions of a defaultable bond are driven by a marked point process $\mu_R(l, dt)$. The marker space of this process is $E = \{1, \dots K\}$ the set of possible rating class transitions. We write the predictable compensator of μ_R with a transition matrix $A = (a_{kl})_{1 \le k, l \le K}$ as

$$\nu_R(l, dt) = \sum_{k=1}^{K} \mathbf{1}_{\{R(t)=k\}} a_{kl} dt = a_{R(t),k} dt.$$

The elements a_{kl} of A can be predictable stochastic processes.

(ii) The dynamics of the defaultable forward rates within class k *are:*

$$d\overline{f}_k(t, T) = \overline{\alpha}_k(t, T)dt + \sum_{i=1}^{n} \overline{\sigma}_{i,k}(t, T)\ dW^i(t) \tag{2.94}$$

(iii) The dynamics of a defaultable pseudo bond price $\widetilde{B}_k(t, T) :=$ $\exp\{-\int_t^T \overline{f}_k(t, s)ds\}$ in rating class k *are:*

$$\frac{d\widetilde{B}_k(t, T)}{\widetilde{B}_k(t-, T)} = m_k(t, T)\ dt + \sum_{i=1}^{n} \overline{a}_{i,k}(t, T)\ dW^i(t) \tag{2.95}$$

(iv) Defaults of the defaultable bond are driven by the multiple default model as in assumption 2.4, definition 2.6 and assumption 2.5. Thus the dynamics of the defaultable bond prices $\overline{B}(t, T) = Q(t)\widetilde{B}_{R(t)}(t, T)$ are:

$$\begin{aligned}\frac{d\overline{B}(t, T)}{\overline{B}(t-, T)} =& m(t, T)\ dt + \sum_{i=1}^{n} \overline{a}_i(t, T)\ dW^i(t) \\ &- q(dN(t) - \lambda(t)dt) + \sum_{k \neq R(t)} \left(\frac{\widetilde{B}_k(t, T)}{\widetilde{B}_{R(t)}(t, T)} - 1 \right) \left(\mu_R(k, dt) - a_{R(t),k} dt\right).\end{aligned} \tag{2.96}$$

(v) *Defaults and rating transitions never happen at the same time:*

$$\int_0^t dN_s \mu_R(k, ds) \equiv 0 \qquad \forall k \in \{1, \ldots, K\},\ t \geq 0.$$

(vi) *We assume sufficient regularity on the parameters to allow:*
– *differentiation (w.r.t. T) under the integral,*
– *interchange of order of integration,*
– *finite prices $\overline{B}(t,T)$, $\widetilde{B}_k(t,T)$ almost surely.*

The ratings transition marked point process $\mu_R(k, dt)$ places Dirac-delta weights of 1 in the space $\{1, \ldots K\} \times I\!R_+$ at points (R^*, t^*), where t^* is the time of a rating transition and R^* is the new rating class.

The compensator matrix A of μ_R contains the intensities of the different rating transitions: a_{kl} is the intensity of the process triggering a transition from class k to class l. A can be stochastic. The matrix A is called the *generator matrix* of the rating transitions. In the theory of Markov chains it is customary to set $a_{kk} := -\sum_{l \neq k} a_{kl}$, here this would not be consistent with the definition of A as predictable compensator of μ_R.

To make the transition intensities a_{kl} stochastic *and* dependent on the original rating class k is redundant, but it makes the simplification to the important case of constant a_{kl} trivial. For a constant rating transition matrix A the ratings transitions form a time-homogeneous Markov chain.

At a rating transition the defaultable bond price jumps to the equivalent (same maturity and face value) defaultable bond price of the new rating class. Similarly, a default leads to a reduction in face value of the defaultable bond by q of the previous face value. The recovery mechanism remains unchanged. Therefore, in equation (2.96) the predictable compensators have already been subtracted from the respective point processes to ensure that the resulting stochastic integrals are local martingales and that $m(t,T)$ contains the full (predictable) drift coefficients. Excluding joint rating transitions and defaults is done mainly to simplify the analysis. This assumption can be relaxed along the lines of the previous section, and it may be desirable to be able to place the bond in a different rating class after a default and reorganisation.

The following proposition gives the connection between the defaultable bond price dynamics, the defaultable pseudo bond price dynamics and the defaultable forward rates for each rating class.

Proposition 2.13
For every given rating class $k = 1, \ldots, K$ and the dynamics (2.94) of $\overline{f}_k(t,T)$

(i) the dynamics of the defaultable short rate $\overline{r}_k(t) = \overline{f}_k(t,t)$ in rating class k are

$$d\overline{r}_k(t) = [\frac{\partial}{\partial T}\overline{f}_k(t,t) + \overline{\alpha}_k(t,t)]dt + \sum_{i=1}^{n} \overline{\sigma}_{i,k}(t,t)dW^i(t). \quad (2.97)$$

(ii) the dynamics of $\widetilde{B}_k(t,T)$ are

$$m_k(t,T) = \overline{r}_k(t) - \int_t^T \overline{\alpha}_k(s,T)\,ds + \frac{1}{2}\sum_{i=1}^{n}(\int_t^T \overline{\sigma}_{i,k}(s,T)\,ds)^2 \quad (2.98)$$

$$\overline{a}_{i,k}(t,T) = -\int_t^T \overline{\sigma}_{i,k}(s,T)\,ds. \quad (2.99)$$

(iii) The dynamics of the defaultable bond prices $\overline{B}(t,T) = \widetilde{B}_{R(t)}(t,T)Q(t)$ and the dynamics of the rating $R(t)$ of this bond are given by assumption 2.7 iv with the following specification:

$$m(t,T) = m_{R(t)}(t,T) - q(t)\lambda(t) + \sum_{k \neq R(t)} \left(\frac{\widetilde{B}_k(t,T)}{\widetilde{B}_{R(t)}(t,T)} - 1 \right) a_{R(t),k}. \quad (2.100)$$

$$\overline{a}_i(t,T) = \overline{a}_{i,R(t)}(t,T) \quad (2.101)$$

$$dR(t) = \sum_{k=1}^{K}(R(t) - k)\mu_R(k,dt). \quad (2.102)$$

Proof: Within a given rating class $k = 1, \ldots, K$ the specification is a standard HJM term structure model of interest rates. Therefore (i) and (ii) follow directly from HJM (1992). Point (iii) follows directly from substituting the respective dynamics into $\overline{B}(t,T) = Q(t)\widetilde{B}_{R(t)}(t,T)$ and using Itô's lemma on point processes.

□

The change of measure to the martingale measure is done according to theorem 2.8. The analysis leading to the key relation (2.77)

$$\lambda(t)q(t) = \overline{r}(t) - r(t) > 0$$

in section 2.5.3 is still valid, but has to be done for each rating class individually.

Consider the rating class k and the investment in the defaultable bank account $c_k(t)$ of this rating class. As the investment in a defaultable bank account always matures at the next instant (and is then rolled over), it is not affected by rating transitions: If our short-term investment had a rating change to class k' (but no default), then it would still have the full payoff in the next instant. This payoff is then taken and invested in a short-term investment in the original rating class k. Thus it is possible to continuously roll over the defaultable bank account $c_k(t)$ in rating class k. It has no jump component in its development except the direct losses of q at default. Therefore (using exactly the same argument as in theorem 2.9) we have

$$\lambda_k(t)q(t) = \overline{r}_k(t) - r(t) > 0, \tag{2.103}$$

the default intensity $\lambda_k(t)$ in rating class k times the loss fraction q equals the short spread in this rating class. The default intensity of the defaultable bond is thus dependent on the current rating class $R(t)$ of the bond.

Equation (2.103) gives a second reason why we did not specify 'default' as $K + 1$st rating class in definition 2.7: Firstly the transition intensity to 'default' is already specified by the term structure of the defaultable bond prices in this class, and secondly, in the fractional recovery model defaulted bonds are reorganised and survive, thus the classical approach (see e.g. Lando (1994; 1998)) of making default an absorbing state is not suitable here.

As usual we need for absence of arbitrage

$$\begin{aligned} r(t)\,dt &= \mathbf{E}\left[\frac{d\overline{B}(t,T)}{\overline{B}(t-,T)}\right] \\ r(t) &= m(t,T). \end{aligned}$$

Substituting (2.103) and the dynamics from proposition 2.13 yields now the following proposition which gives the drift restrictions that are to be imposed on the martingale-measure dynamics of the defaultable forward rates of each rating class to ensure absence of arbitrage of the full model.

Proposition 2.14

Under the martingale measure

(i) The short rate spread in rating class k ($k \in \{1, \dots K\}$) is given by

$$\lambda_k(t)q(t) = \overline{r}_k(t) - r(t). \tag{2.104}$$

(ii) The drift of the defaultable forward rates is restricted by

$$\int_t^T \overline{\alpha}_k(t,s)ds = \frac{1}{2}\sum_{i=1}^{n}(\int_t^T \overline{\sigma}_{i,k}(t,s)\,ds)^2 + \sum_{l=1}^{K}\left(\frac{\widetilde{B}_l(t,T)}{\widetilde{B}_k(t,T)} - 1\right) a_{k,l} \tag{2.105}$$

or differentiated:

$$\overline{\alpha}_k(t,T) = \sum_{i=1}^{n}\overline{\sigma}_{i,k}(t,T)(\int_t^T \overline{\sigma}_{i,k}(t,s)\,ds) + \sum_{l=1}^{K}\frac{\widetilde{B}_l(t,T)}{\widetilde{B}_k(t,T)}(\overline{f}_k(t,T) - \overline{f}_l(t,T))\,a_{k,l}. \tag{2.106}$$

If the defaultable forward rates satisfy (2.105) *or* (2.106)*, discounted defaultable bond prices are local martingales.*

This result holds without assuming Markov chain dynamics for the rating transition process, and it can easily be generalised to stochastic recovery rates and joint jumps of ratings and defaults. The drift restrictions (2.105) and (2.106) are versions of the drift restriction found by BKR in the context of default-free term structures of interest rates. To close the model, the default-free forward rates have to satisfy the classical HJM-drift restrictions, too.

Using proposition 2.14 it is now possible to set up a simulation model of defaultable bond price dynamics for a full portfolio of defaultable bonds of different rating classes and maturities. Correlations in the movements of the spreads and yields in all classes and in the dynamics of the default-free term structure of interest rates can be captured directly via the forward rate volatility functions $\overline{\sigma}_k(t,T)$.

The only remaining problem is the specification of the rating transition intensities a_{kl} under the *martingale measure*. There is a wealth of published data concerning *historical* rating transitions but there seems to be a large risk-premium in the market attached to downgrades below investment-grade and to defaults. This problem is left for further research.

2.9 Conclusion

In this chapter we presented a new approach to the modelling of the price processes of defaultable bonds that was inspired by the Heath-Jarrow-Morton (1992) model of the term structure of interest rates. This model avoids a precise specification of the mechanism that leads to default but rather gives necessary and sufficient conditions on the term structure of defaultable interest rates to ensure absence of arbitrage.

These restrictions show a striking similarity to the restrictions that are already well known from default-free term structure models. Specifically, the defaultable interest rates have to satisfy a drift restriction that is analogous to the HJM drift restriction, and a positive short spread restriction. Given these restrictions the model is arbitrage-free. In the implementation of the model one can therefore use the extensive machinery of default-free interest rate modelling.

As an alternative to the HJM-modelling approach it is shown that sufficient for the absence of arbitrage in a short rate model is a positive spread between the defaultable and the default-free short rate. Furthermore it is discussed how a defaultable bond model can be added to an existing model of default-free bonds while keeping the combined model arbitrage-free, and the key result of the direct intensity approach was recovered, viz the representation of the defaultable bond prices as the expectation of the defaultable discount factor.

In addition to this, a new *multiple default model* was introduced in this chapter in which defaulted firms are reorganised and their bonds continue to trade after a default. It was shown that this modelling approach leads to a very intuitive representation of the spread of the defaultable short rate over the default-free short rate as the product of the loss quota in default and the default intensity.

Finally, in the last two sections the approach was extended in two directions: First, jumps were allowed in the term structure of defaultable interest rates at times of default, and second, the model was extended to a full rating transition model. The rating transition model extended the existing literature in several ways: It is the first default model to allow: fully independent stochastic dynamics of the credit spreads in all rating classes, a perfect fit to the term structures of credit spreads in all rating classes, and stochastic rating transition matrices.

Chapter 3

The Pricing of Credit Risk Derivatives

In this chapter the pricing of several credit risk derivatives is discussed in an intensity-based framework with both default-free and defaultable interest rates stochastic and possibly correlated. The differences to standard interest rate derivatives are analysed. Where possible closed-form solutions are given.

3.1 Introduction

Credit derivatives are derivative securities whose payoff depends on the credit quality of a certain issuer. This credit quality can be measured by the credit rating of the issuer or by the yield spread of his bonds over the yield of a comparable default-free bond. In this chapter we will concentrate on the latter case.

Credit risk derivatives can make large and important risks tradeable. They form an important step towards market completion and efficient risk allocation, they can help bridge the traditional market segmentation between corporate loan and bond markets.

Despite their large potential practical importance, there have been very few works specifically on the pricing and hedging of credit risk *derivatives*. The majority of papers is concerned with the pricing of defaultable *bonds* which is a necessary prerequisite for credit derivatives pricing. In this chapter we are going to use mainly the fractional recovery / multiple defaults model (see Duffie and Singelton (1997; 1999) and also Schönbucher (1996a; 1998)) which we will compare to an alternative model of the recovery of defaulted bonds: recovery in terms of otherwise equivalent default-free bonds (here termed *equivalent recovery*, see e.g. Lando (1994; 1998)). This will enable us to analyse the relative advantages and disadvantages of the two modelling approaches.

Apart from this comparison, the second main point of the chapter is to demonstrate the methodology that has to be used in an intensity-based credit spread model to price credit contingent payoffs. This methodology is independent of the actual specification of the model and it is easy to transfer most of the results to other intensity-based models like e.g. the models by Jarrow and Turnbull (1995), Lando (1994; 1998), Madan and Unal (1998) or Artzner and Delbaen (1992). Closely related to this point is the third aim of the chapter which is to analyse and identify the relative importance of model input parameters like recovery rates or the correlation between interest rate and credit spreads.

In the following section the structure and applications of the most popular credit derivatives will be presented. As credit derivatives are over-the-counter derivatives, no standard specification has evolved yet. Therefore we chose an wide array of common and/or natural specifications to exemplify typical specifications and applications, and the more exotic specifications will not be treated in the section on pricing later on. The discussed securities include *total rate of return swaps*, *default swaps*, straight *put options* on defaultable bonds, *credit spread forwards* and *credit spread puts*.

This is followed by a recapitulation of the fractional recovery and the equivalent recovery models and a short discussion of the relative advantages and disadvantages of both approaches. Using these models we begin the analysis of the pricing of those credit derivatives that can be priced directly off the term structures of interest rates and credit spreads, without needing to explicitly specify the dynamics of these term structures.

In the next section we present the specification of the stochastic processes for interest rates and credit spreads/default intensities. We chose two alternative setups for the credit spread and for the default-free short rate, a multifactor Gaussian model and a multifactor Cox Ingersoll Ross (CIR) (1985b) square-root diffusion model.

The Gaussian setup can allow for arbitrary correlation between credit spreads and the default-free term structure of interest rates but has the disadvantage of allowing credit spreads and intensities to become negative. With the CIR setup on the other hand, only positive correlation can be reached but it ensures positive interest rates and credit spreads. Obviously these are not the only possible specifications, another promising specification would be a market model with lognormal LIBOR rates. (See Miltersen, Sandmann and Sondermann (1997) for the default-free case and Lotz and Schloegl (1999) for an analysis of mutual counterparty risk in a market model.)

The next section treats the pricing of the most important of the the credit risk derivatives that were introduced before. To keep the results as flexible as possible we used the specifications only in the last step of the calculation of the prices, and we also give the expectations that have to be calculated if another model is used or a numerical implementation is desired. In the CIR case the pricing formulae are expressed in terms of the multivariate noncentral chi-squared distribution function using results from Jamshidian (1996), in the Gaussian case we reach expressions in terms of the cumulative normal distribution function. We analyse the influence of interest-rate / credit spread correlation and the differences to standard interest rate contingent contracts.

The conclusion sums up the main results of the chapter and points out the consequences that these results have for the further development of credit spread models.

3.2 Credit Derivatives: Structures and Applications

This section contains an overview over the most common credit derivatives. First we have to clarify the common features of most credit derivatives.

Definition 3.1
A credit derivative *is a derivative security that has a payoff which is conditioned on the occurrence of a* credit event. *The credit event is defined with respect to a* reference credit *(or several reference credits), and the* reference credit asset(s) *issued by the reference credit. If the credit event has occurred, the* default payment *has to be made by one of the counterparties.*
Besides the default payment a credit derivative can have further payoffs that are not default contingent.

Most credit derivatives have a default-insurance feature. In naming the counterparties we will use the convention that counterparty **A** will be the insured counterparty (i.e. the counterparty that receives a payoff if a default happens or the party that is long the credit derivative), and counterparty **B** will be the insurer (who has to pay in default). Party **C** will be the reference credit.

3.2.1 Terminology

One of the attractions of credit derivatives is the large degree of flexibility in their specification. Key terms of most credit derivatives are:

Reference Credit: One (or several) issuer(s) whose defaults trigger the credit event. This can be one (typical) or several (a basket structure) defaultable issuers.

Reference Credit Asset: A set of assets issued by the reference credit. They are needed for the determination of the credit event and for the calculation of the recovery rate (which is used to calculate the default payment).
The definition can range from 'any financial obligation of the reference credit' to a specific list of just a few bonds issued by the reference credit. Loans and liquidly traded bonds of the reference credit are a common choice. Frequently, different assets are used for the determination of the credit event and the recovery rate[1].

Credit Event: A precisely defined default event, which is usually defined with respect to the reference credit(s) and reference credit assets.
Possible definitions include:

- payment default (typically a certain materiality threshold must be exceeded)
- bankruptcy or insolvency,
- protection filing,
- ratings downgrade below given threshold (ratings triggered credit derivatives),
- changes in the credit spread
- payment moratorium

but the definition can include events that go as far as 'armed hostilities', 'social unrest' or earthquakes (for souvereigns) or 'merger or takeover' (for corporates).

Default Payment: The payments which have to be made if a credit event has happened. The default payment is the defining feature of most credit derivatives.
This is the defining feature of most credit derivatives, and we will consider possible alternatives when discussing the individual credit derivatives.

Example 3.1:

Default digital swap on the United States of Brazil:

[1] Assume a bank has a large loan exposure to **C**. To hedge this, a credit derivative could use a missed payment on the *loan* as credit event trigger and the post-default market price of a *bond* issued by **C** to determine the recovery rate. As the loan is not traded, its recovery rate cannot be determined from market prices.

Counterparty **B** (the insurer) agrees to pay USD 1 Mio to counterparty **A** if and when Brazil misses a coupon or principal payment on one of its Eurobonds. Here

- the *reference credit* are the United States of Brazil
- the *reference credit assets* are the Eurobonds issued by Brazil (in the credit derivative contract there would be an explicit list of these bonds),
- the *credit event* is a missed coupon or principal payment on one of the reference assets,
- the *default payment* is USD 1 Mio.

In return for this, counterparty **A** pays a fee to **B**.

So far no standard has evolved in the details of the specification of credit derivatives and there are many unresolved problems in this area. Despite the large degree of flexibility it is impossible to cover every contingency in the definition of the credit event, and – apart from digital payoffs – there will also be problems to match the default payment exactly to the exposure that is to be hedged. Even with physical delivery there may be problems if the reference asset is very illiquid or not traded at all (e.g. a loan).

Ignoring the problems in the details of the specification, some credit derivatives have become quasi-standard credit derivative structures. In the following sections we will examine these structures and their applications in further detail.

3.2.2 Asset Swap Packages

An *asset swap package* is a combination of a defaultable fixed coupon bond (the asset) with a fixed-for-floating interest rate swap whose fixed leg is chosen such that the value of the whole package is the par value of the defaultable bond.

Example 3.2:

The payoffs of the asset swap package are:

B sells to **A** for 1 (the nominal value of the **C**-bond):

- a fixed coupon bond issued by **C** with coupon c payable at coupon dates $t_i,\ i = 1, \ldots, N$,
- a fixed for floating swap (as below).

The payments of the swap: At each coupon date $t_i,\ i \leq N$ of the bond

- **A** pays to **B**: c, the amount of the fixed coupon of the bond,
- **B** pays to **A**: LIBOR + a.

a is called the *asset swap spread* and is adjusted to ensure that the initial asset swap package has indeed the value of 1.

The asset swap is not a credit derivative in the strict sense, because the swap is unaffected by any credit events. Its main purpose is to transform the payoff streams of different defaultable bonds into the same form: *LIBOR + asset swap spread* (given that no default occurs). **A** still bears the full default risk and if a default should happen, the swap would still have to be serviced.

To ensure that the value of the asset swap package (asset swap plus bond) to **A** is at par at time t_0 we require:

$$C_0 + (s_0 + a_0 - c)A_0 = 1 \tag{3.1}$$

where C_0 is the initial price of the bond, s_0 is the fixed-for-floating swap rate for the same maturity and payment dates t_i, and A_0 is the value of an annuity paying 1 at all times $t_i,\ i = 1, \ldots, N$. All these quantities can be readily observed in the market at time t_0. To ensure that the value of the asset-swap package is indeed one, the asset swap rate must be chosen as

$$a_0 = \frac{1}{A_0}(1 - C_0) + c - s_0.$$

Note that the asset swap rate would explode at a default of **C**, because then $(1 - C_i)$ would change from being very small to a large number. Using the definition of the fixed-for-floating swap rate: $s_0 A_0 = 1 - B(t_0, t_N)$ this can be rearranged to yield:

$$A_0 a_0 = \underbrace{B(t_0, t_N) + cA_0}_{\text{def. free bond}} - \underbrace{C_0}_{\text{defaultable bond}}, \tag{3.2}$$

the asset swap rate a_0 is the price difference between the defaultable bond C_0 and an equivalent default free coupon bond (with the same coupon c, it has the price $B(t_0, t_N) + cA_0$) in the swap-measure numeraire asset A_0.

Asset swap packages are very popular and liquid instruments in the defaultable bonds market, sometimes their market is even more liquid than the market for the underlying defaultable bond alone. They also serve frequently as underlying assets for options on asset swaps, so called *asset swaptions*. An asset swaption gives **A** the right to enter an asset-swap package at some future date T_1 at a pre-determined asset swap spread a.

3.2.3 Total Rate of Return Swaps

In a *total rate of return swap* (or *total-return swap*) A and B agree to *exchange all cash flows* that arise from two different investments, usually one of these two investments is a defaultable investment, and the other is a default-free LIBOR investment. This structure allows an exchange of the assets' payoff profiles without legally transferring ownership in the asset.

Example 3.3:
Payoffs of a total rate of return swap: Counterparty **A** pays to counterparty **B** at regular intervals:

- the coupon of the bond **C** (if there was one)
- the price appreciation of bond **C** since the last payment
- the principal repayment of bond **C** (at the final payment date)
- the recovery value of the bond (if there was a default)

B pays at the same intervals

- a regular fee of LIBOR $+x$
- the price depreciation of the bond **C** (if there was any)
- the par value of the bond (if there was a default in the meantime)

These payments are netted.

In the example above the two investments whose payoff streams are exchanged are:
(a) an investment of a dollar amount of the face value of the **C** bond at LIBOR and (b) the investment in the **C** bond, adjusted by a spread on the LIBOR investment. The reference credit is **C**, the reference asset is the **C** bond, credit event is a default on the reference asset and the payoff in default is specified above.

B has almost the same payoff stream as if he had invested in the bond **C** directly and funded this investment at LIBOR $+x$. The only difference is that in the total rate of return swap is marked to market at regular intervals. Price changes in the bond **C** become cash flows for the TRORS immediately, while for a direct investment in the bond they would only become cash flows when the bond matures or the position is unwound. This makes the TRORS similar to a futures contract on the **C** bond, while the direct investment is more similar to the forward. The TRORS is not exactly equivalent to a futures contract because it is marked to market using the *spot* price of the underlying security, and not the *futures* price. This difference can be adjusted using the spread x on the floating payment of **B**.

The reference asset should be liquidly traded to ensure objective market prices for the marking to market. If this is not the case (e.g. for bank loans), the total return swap cannot be marked to market. Then its term must match the term of the underlying loan or it must be terminated by physical delivery.

Total rate of return swaps are among the most popular credit derivatives. They has several advantages to both counterparties:

- Counterparty **B** is long the reference asset without having to fund the investment up front. This allows counterparty **B** to leverage his position much higher than he would otherwise be able to.
- If the reference asset is a loan and **B** is not a bank then this may be the only way in which **B** can invest in the reference asset.
- Counterparty **A** has hedged his exposure to the reference credit if he owns the reference asset (but he still retains some counterparty risk).
- If **A** does not own the reference asset he has a created short position in the asset. (Directly shorting defaultable bonds or loans is often impossible.)
- The transaction can be effected without the consent or the knowledge of the reference credit **C**. **A** is still the lender to **C** and keeps the bank-customer relationship.

3.2.4 Default Swap

In a *default swap* (also known as *credit swap*) **B** agrees to pay the default payment to **A** *if a default has happened.* If there is no default of the reference security until the maturity of the default swap, counterparty **B** pays nothing.

A pays a fee for the default protection. The fee can be either a lump-sum fee up front (default put) or a regular fee at intervals until default or maturity (default swap).

An example of a default swap with a fixed repayment at default is given in example 1 (default digital swap on Brazil). Default swaps mainly differ in the specification of the default payment. Common alternatives are

- notional minus post-default market value[2] of the reference asset (cash settlement),

[2] The post-default market value could be determined from dealer bid and ask quotes, averaged over a certain period of time and over several dealers and reference assets.

- physical delivery of one or several of the reference assets against repayment at par,
- a pre-agreed fixed percentage of the notional amount (default digital swap).

Sometimes substitute securities may be delivered, or an exotic payoff may be specified (e.g. to hedge counterparty exposure in derivatives transactions).

The default swap allows the separation of the credit risk component of a defaultable bond from its non-credit driven market risk components. The protection buyer (**A**) retains the market risk but is hedged against the credit risk of **C**, while the protection seller (**B**) can assume the credit risk alone.

The price of a default swap is closely related to the price of a *defaultable floating rate note (FRN)*. Although the defaultable FRN is not a derivative, its pricing is nevertheless nontrivial because it does not always have to trade at par like default-free FRNs. This relationship will be explained in the section on the pricing of credit derivatives.

3.2.5 Credit Spread Products

Some credit derivatives have payoffs that condition on the credit spread of the reference credit asset over an equivalent default-free bond. Here the credit event is a change in the credit spread and not necessarily a default. To isolate the reference on credit spreads some credit spread derivatives are even knocked out if a default happens on the reference asset. The reference asset must be a liquidly traded bond to allow a meaningful definition of the credit spread.

Credit spread structures are mainly used for trading. They allow the counterparties to separately trade the credit risk component of a defaultable bond and contribute thus to more efficiency in these markets. A credit spread forward can be used to make a position in the reference asset neutral against credit spread movements or defaults, and a credit spread put can be used to limit the downside risk due to credit spread movements. With a very high strike spread a credit spread put can also be used as a substitute for a default swap which can have advantages for regulatory purposes.

3.2.5.1 Credit Spread Forward and Credit Spread Swap

In a *credit spread forward*, counterparty **A** pays at time T a pre-agreed fixed payment and receives the credit spread of the reference asset at time T. Conversely, counterparty **B** receives the fee and pays the credit spread. The fixed payment is chosen at time $t < T$ to set the initial value of the credit spread forward to zero.

The credit spread forward can also be structured around the *relative* credit spread between two different defaultable bonds. This case can be decomposed into two credit spread forwards on the *absolute* credit spread of the reference assets. The credit spread forward is similar to a forward rate agreement with the only difference that the credit spread is referenced and not an interest rate. Several credit spread forwards can be combined to a *credit spread swap*.

3.2.5.2 Credit Spread Options

A *credit spread put* gives counterparty **A** the right to sell the reference asset to counterparty **B** at a pre-specified strike spread over default-free interest rates. A credit spread put can be viewed as an exchange option that gives **A** the right to exchange one defaultable bond for a certain number (< 1) of default free bonds. Frequently the underlying security is not a defaultable bond but an asset swap package on the defaultable bond.

- the *reference credit asset* is the referenced bond or asset swap
- the *credit event* has occured if the credit spread of the reference asset is above the strike spread at maturity of the option
- the *default payment* is the price that the reference asset would have at the strike spread minus the market price of the reference asset.

Many funds and insurance companies are restricted by their statutes or regulation to investment-grade investments. A credit spread put option would allow these investors a switch out of the defaultable bond investment if the credit quality of the reference credit decreases. For this application the credit event could also be defined as a downgrade of the reference credit to a rating class below investment grade, and the default payment can be an exchange of the reference asset against an index of investment-grade debt.

A second application is the exposure management of committed lines of credit. A committed line of credit is similar to an overdraft on a bank account. The debtor **C** has the right to enter a pre-agreed loan contract at any time he chooses, he can draw his line of credit. For this **C** pays a regular fee to the committed bank **A**. If bank **A** does not want any additional exposure to **C**, it can still commit the line of credit, but hedge with a credit spread option (American style) which enables bank **A** to put the loan to **B** as soon as the line of credit is drawn[3]. In this context the option is also called a *synthetic lending facility*.

3.2.6 Options on Defaultable Bonds

A plain *put option* on a defaultable bond gives **A** the right to sell the defaultable bond to **B** at the strike price at the exercise date. It is different from a classical bond option in that the price of the underlying (the defaultable bond) may jump downwards at defaults. It offers protection against rising default-free interest rates, rising credit spreads and (if the option survives) against defaults.

3.2.7 Basket Structures

In a *basket default swap* we have several reference credits $C_1, \ldots, C_N$ with their respective reference credit assets. The credit event can be triggered either by the first default in the basket (first-to-default) or if a certain loss level is exceeded. A first-to-default basket default swap would have the following payoffs:

- **A** pays a regular fee.
- The credit event is the first default of one of the reference credits.
- If the credit event has happened, **B** pays to **A** the default payment.
- The default payment is either par minus the recovery value of the defaulted security, or a binary payment, or another default payment.
- After the first default the basket default swap is terminated.

With a first-to-default swap the credit quality of a portfolio can be enhanced significantly (provided the defaults in the portfolio are largely uncorrelated). **A**'s motivation here is to gain default protection at a lower price while accepting the risk of more than one default in the basket. For counterparty **B** a basket structure can have regulatory capital advantages[4]. Furthermore **B** can reach a higher (promised) return on its investment than he could by investing in one of the reference credits alone.

Often, the combination with a basket default swap can enhance the credit quality of a loan portfolio to 'investment grade' rating and thus make it suitable for a new class of investors. The first-to-default structure is most useful for baskets with a small number (less than six)

[3] A priori this would not be an optimal early exercise policy, but given bank **A**'s operating constraints it may still be optimal to them.

[4] By the BIS regulatory capital rules **B** only has to provide regulatory capital for *one* defaultable bond, although he bears most of the credit risk of the whole basket.

of defaultable bonds, for large portfolios it is more appropriate to specify a certain loss level that has to be exceeded.

3.2.8 Credit Linked Notes

A *credit linked note* is combination of a credit derivative (usually a default swap or a basket default swap) with a bond issued by counterparty **A**. A default-swap linked note has the following structure

- **B** buys the note.
- At the coupon dates **A** pays the coupon of the note (provided **C** has not defaulted).
- If **C** defaults, the note is terminated and **A** pays the recovery rate of the reference asset (a bond issued by **C**) to **B**.
- If **C** has not defaulted until maturity of the note, **A** repays the principal amount of the note.

In general, after a default of **C**, **A** pays to **B** the nominal value of the note *minus* the default payment of the credit derivative that is combined with the note. To counterparty **B** this structure is (almost[5]) equivalent to an investment in a bond issued by **A** and a short position in the credit derivative with **C** as reference credit.

In such a structure, **A** is often a bank that has given a loan to **C**. **A**'s motives are the following:

- **A** receives funding for the loan to **C**.
- **A** has laid off the credit risk to **C**.
- There is no counterparty risk to **A** because **A**'s claims are fully collateralised. (On the other hand **B** does bear counterparty risk to **A**.)
- The credit protection can be bought more easily: Counterparty risk is eliminated and the note can be sold in small denominations to more than one investors.

To **B** this credit linked note offers customised exposure to **C**'s credit risk even if **B** himself would not qualify as a default swap counterparty because of his credit standing. If **A** has a very high credit rating, the credit linked note is equivalent to debt issued by **C** directly. Total return credit linked notes are also a popular vehicle for the securitisation of large pools of small claims in form of *collateralised loan obligations (CLO).*

[5] The only difference is that after a default of **C** the credit linked note is terminated while a bond issued by **A** would survive.

3.2.9 Applications of Credit Derivatives

Some of the applications of credit derivatives have already been mentioned when they were specific to the credit derivative discussed. General fields of application common to most credit derivatives are:

- Applications in the management of credit exposures:
  These include the reduction of credit concentration (through basket structures), easier diversification of credit risk and the direct hedging of default risk.
- In trading, credit derivatives can be used for the arbitrage of mispricing in defaultable bonds (through the possibility of short positions on credit risk) and the general possibility to trade a view on the credit quality of a reference credit (usually through credit spread products). Default digital products also allow the trading of views on the recovery rate of defaulted debt.
- The largest group of credit derivative users are banks who use credit derivatives to free up or manage credit lines, manage loan exposure without needing the consent of the debtor, manage (or arbitrage) regulatory capital or exploit comparative advantages in costs of funding. Another important application here is the securitisation of loan portfolios in form of CLOs.
- The specification of the credit derivatives can be adjusted to the needs of the counterparties: Denomination, currency, form of coupon, maturity or even the general payoff need not match the reference asset. This is especially useful for the management of counterparty exposures from derivatives transactions.

3.3 Defaultable Bond Pricing with Cox Processes

3.3.1 Model Setup and Notation

As in the previous chapter, the model is set up in a filtered probability space $(\Omega,\ \mathcal{F}, (\mathcal{F}_t)_{(t\geq 0)}, P)$ where P is a pre-specified martingale measure. We assume the filtration $(\mathcal{F}_t)_{(t\geq 0)}$ satisfies the usual conditions[6] and the initial filtration $\mathcal{F}_0$ is trivial. We also assume a finite time horizon $\bar{T}$ with $\mathcal{F} = \mathcal{F}_{\bar{T}}$, all definitions and statements are understood to be only valid until this time horizon $\bar{T}$.

Unless otherwise stated the notation follows the specification in the previous chapter and

[6] See Jacod and Shiryaev (1988).

the notation overview behind the table of contents. We will use the definition 2.2 on page 14 of the default-free term structure, and definition 2.3 on page 15 for the defaultable term structure of bond prices and interest rates. The notation used is:

- $B(t,T)$: default free zero coupon bond price,
- $r(t)$: default free short rate,
- $\beta_{t,T}$: discount factor over $[t,T]$,
- $\overline{B}(t,T)$: defaultable bond price,
- $P(t,T)$: survival probability for $[t,T]$.

3.3.2 The Time of Default

Although we are going to use two different models to model the *recovery* of defaulted bonds, the model for the *time* of the default(s) is the same for both:

We assume that the times of default τ_i are generated by a Cox process. Intuitively, a Cox Process is defined as a Poisson process with stochastic intensity λ (see Lando (1998), p.101). Formally the definition is:

Definition 3.2
N is called a Cox process, if there is a nonnegative adapted stochastic process $\lambda(t)$ (called the intensity *of the Cox process) with $\int_0^t \lambda(s)ds < \infty \quad \forall\ t > 0$, and conditional on the realization $\{\lambda(t)\}_{\{t>0\}}$ of the intensity, $N(t)$ is a time-inhomogeneous Poisson process with intensity $\lambda(t)$.*

This definition follows Lando (1998) and differs from the usual definition of a Cox process where the intensity process $\lambda(t)$ is $\mathcal{F}_0$-measurable (see e.g. Brémaud (1981)). It is not necessary to reveal directly *all* information about the future development of the intensity, and for the valuation of some derivatives this modelling approach would even introduce pricing errors[7].

Assumption 3.1
(i) The default counting process

$$N(t) := \max\{i|\tau_i \le t\} = \sum_{i=1}^{\infty} \mathbf{1}_{\{\tau_i \le t\}} \tag{3.3}$$

[7] Consider e.g. an American Put option on a defaultable bond in a world with constant zero default-free interest rates. If all information about $\lambda(t)$ is revealed at $t = 0$ this would enable the investor to condition his optimal exercise policy on the future development of λ which is not realistic.

is a Cox process with intensity process $\lambda(t)$.

(ii) In the equivalent recovery model the time of default is the time of the first jump of N. *To simplify notation the time of the* first *default will be referred to with* $\tau := \tau_1$.

(iii) In the fractional recovery model the times of default are the times of the jumps of N.

Remark 3.1

Given the realisation of λ, the probability of having exactly n jumps is

$$\mathbf{P}\left[\, N(T) - N(t) = n \mid \{\lambda(s)\}_{\{T \geq s \geq t\}} \,\right] \;=\; \frac{1}{n!}\left(\int_t^T \lambda(s)ds\right)^n \exp\{-\int_t^T \lambda(s)ds\}. \quad (3.4)$$

The probability of having n jumps (without knowledge of the realisation of λ) is found by conditioning on the realisation of λ *within* an outer expectation operator:

$$\begin{aligned} \mathbf{P}\left[\, N(T) - N(t) = n \mid \mathcal{F}_t \right] &= \mathbf{E}\left[\; \mathbf{P}\left[\, N(T) - N(t) = n \mid \{\lambda(s)\}_{\{T \geq s \geq t\}} \,\right] \;\middle|\; \mathcal{F}_t \right] \\ &= \mathbf{E}\left[\; \frac{1}{n!}\left(\int_t^T \lambda(s)ds\right)^n \exp\{-\int_t^T \lambda(s)ds\} \;\middle|\; \mathcal{F}_t \right], \quad (3.5) \end{aligned}$$

Define the process $P(t,T)$

$$P(t,T) = \mathbf{E}\left[\, e^{-\int_t^T \lambda(s)ds} \;\middle|\; \mathcal{F}_t \right]. \quad (3.6)$$

For $\tau > t$ $P(t,T)$ can be interpreted as the *survival probability* from time t until time T. In general,

$$\mathbf{1}_{\{\tau > t\}} P(t,T) = \mathbf{E}\left[\, \mathbf{1}_{\{\tau > T\}} \mid \mathcal{F}_t \right]. \quad (3.7)$$

Given $\tau > t$ the density of the time of the first default as seen from t is for $T > t$

$$p(t,T) = \mathbf{E}\left[\, \lambda(T) \exp\{-\int_t^T \lambda(s)ds\} \;\middle|\; \mathcal{F}_t \right], \quad (3.8)$$

and $p(t,T) = 0$ for $T \leq t$. [8]

The law of iterated expectations will prove extremely useful later on, it was first used with Cox processes in a credit risk context by Lando (1998).

[8] If a default has already happened, $p(t,T) = \epsilon_\tau$ the density of the first default reduces to the Dirac measure at τ.

The specification of the default trigger process as a Cox process precludes a dependence of the default intensity on previous defaults[9] and also ensures totally inaccessible stopping times τ_i as times of default. Apart from this it allows rich dynamics of the intensity process, specifically, we can reach stochastic credit spreads. If only the time of the *first* jump of N is of interest, the Cox-process specification is completely without loss of generality within the totally inaccessible stopping times.

Equations (3.4), (3.5) and (3.8) will be used frequently later on. In the following sections we will consider time t as 'today', and assume that no default has happened so far $\tau > t$. (The statements for $\tau < t$ are trivial.)

3.3.3 The Fractional Recovery Model

The model used here is an extension of the Duffie-Singleton (1999) model to multiple defaults. More details to the model can be found in Schönbucher (1996a; 1998) and in section 2.5 in the previous chapter. The new feature of this model is that a default does not lead to a liquidation but a reorganisation of the issuer: defaulted bonds lose a fraction q of their face value and continue to trade. This feature enables us to consider European-type payoffs in our derivatives without necessarily needing to specify a payoff of the derivative at default (although we will consider this case, too). The next assumption (a repetition of assumption 2.4) summarises the fractional recovery model:

Assumption 3.2
There is an increasing sequence of stopping times $\{\tau_i\}_{i\in\mathbb{N}}$ that define the times of default (given in definition 3.2 and assumption 3.1). At each default τ_i the defaultable bond's face value is reduced by a factor q_i, where q_i may be a random variable itself. A defaultable zero coupon bond's final payoff is the product

$$Q(T) := \prod_{\tau_i \le T} (1 - q_i) \tag{3.9}$$

of the face value reductions after all defaults until the maturity T of the defaultable bond. The loss quotas q_i can be random variables drawn from a distribution $K(dq)$ at time τ_i, but for the first calculations we will assume $q_i = q$ to be constant.

[9] It is therefore not possible to specify an intensity that jumps at defaults.

It is now easily seen[10] that in this setup the price of a defaultable zero coupon bond is given by

$$\overline{B}(t,T) = Q(t)\mathbf{E}\left[e^{-\int_t^T \overline{r}(s)ds} \;\middle|\; \mathcal{F}_t \right]. \tag{3.10}$$

The process $\overline{r}$ is called the defaultable short rate $\overline{r}$ and it is defined by

$$\overline{r} = r + \lambda q. \tag{3.11}$$

Here r is the default-free short rate, λ the hazard rate of the defaults and q is the loss quota in default. If q is stochastic then q has to replaced by its (local) expectation $q_t^e = \int qK_t(dq)$ in equation (3.11).

It is convenient to decompose the defaultable bond price $\overline{B}$ as follows:

$$\overline{B}(t,T) = Q(t)\widetilde{B}(t,T) = Q(t)B(t,T)\widetilde{P}(t,T). \tag{3.12}$$

Here $Q(t)$ represents the face-value reduction due to previous defaults (before time t). Frequently we will be able to set $t = 0$ and thus $Q(t) = 1$, but for the analysis at intermediate times it is important to be clear about the notation[11]. The defaultable bond price $\overline{B}(t,T)$ is thus the product of $Q(t)$, the influence of previous defaults, and the product of the default-free bond price $B(t,T)$ and the third factor $\widetilde{P}(t,T)$ which is uniquely defined by equation (3.12), or equivalently:

$$\widetilde{P}(t,T) = \frac{1}{Q(t)}\frac{\overline{B}(t,T)}{B(t,T)}. \tag{3.13}$$

Remark 3.2

$\widetilde{P}(t,T)$ is related to the *survival probability* $P(t,T)$ of the defaultable bond: If r and λ are independent and there is a total loss ($q = 1$) at default then $\widetilde{P}(t,T)$ is the probability (under the martingale measure) that there is no default in $[t,T]$.

If r and λ are not independent, $\widetilde{P}(t,T)$ is the survival probability under the T-forward measure P^T

$$\widetilde{P}(t,T) = \mathbf{E}^{P^T}\left[e^{-\int_t^T q\lambda(s)ds} \;\middle|\; \mathcal{F}_t \right]. \tag{3.14}$$

(Under independence $\mathbf{E}^T\left[e^{-\int_t^T q\lambda(s)ds} \;\middle|\; \mathcal{F}_t \right]$ and $\mathbf{E}\left[e^{-\int_t^T q\lambda(s)ds} \;\middle|\; \mathcal{F}_t \right]$ coincide.)

[10] Using the iterated expectations, see equation (2.81) and also Duffie and Singleton (1999) and Schönbucher (1996a; 1998) for a more general proof.

[11] For example, at the expiry date of an option we would like to separate previous defaults and credit spreads in the price of the underlying.

If there is positive recovery ($q < 1$) then $\widetilde{P}(t,T)$ is the *expected final payoff* under the T-forward measure, but the implied survival probability cannot be recovered.

3.3.4 The Equivalent Recovery Model

The equivalent recovery model has been proposed by several authors, amongst them Jarrow and Turnbull (1995) Lando (1998) and Madan and Unal (1998). Here the recovery of defaulted debt is treated as follows:

Assumption 3.3
At the time of default τ one defaultable bond $\overline{B}(\tau,T)$ with maturity T has the payoff of c default free bonds $B(\tau,T)$ of the same maturity and face value, where c may be random, too.

Under the equivalent recovery model (with constant c and given no default so far $\tau > t$) the price of a defaultable bond can be decomposed into c default-free bonds and $(1-c)$ defaultable bonds with zero recovery

$$\begin{aligned}\overline{B}(t,T) &= \mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} + c\beta_{t,\tau}B(\tau,T)\mathbf{1}_{\{\tau\le T\}} \;\middle|\; \mathcal{F}_t \,\right] \\ &= \mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \;\middle|\; \mathcal{F}_t \,\right] + c\mathbf{E}\left[\, \beta_{t,T} \;\middle|\; \mathcal{F}_t \,\right] - c\mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \;\middle|\; \mathcal{F}_t \,\right] \\ &= (1-c)\overline{B}_0(t,T) + cB(t,T), \end{aligned} \tag{3.15}$$

where $\overline{B}_0(t,T)$ is the price of a defaultable bond under zero recovery:

$$\mathbf{E}\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}} \;\middle|\; \mathcal{F}_t \,\right] = \mathbf{1}_{\{\tau>t\}}\mathbf{E}\left[\, e^{-\int_t^T r(s)+\lambda(s)ds} \;\middle|\; \mathcal{F}_t \,\right].$$

It should be pointed out that the equivalent recovery model is not able to fit all term structures of credit spreads with a given fixed common recovery rate c. Assume $\tau > t$ and the term structure of credit spreads is at a constant credit spread h for all maturities T. Then

$$\frac{\overline{B}(t,T)}{B(t,T)} = e^{-h(T-t)}$$

and for large enough $T-t$ (such that $T-t > -(\ln c)/h$),

$$\widetilde{P}(t,T) = \frac{1}{1-c}\left(\frac{\overline{B}(t,T)}{B(t,T)} - c\right) = \frac{1}{1-c}\left(e^{-h(T-t)} - c\right) < 0,$$

the survival probability (see below) that can be implied from the zero-recovery bond $\overline{B}_0(t,T)$ would become negative, which is obviously not sensible. In the equivalent recovery model there is a lower bound on the ratio of defaultable bond prices to default-free bond prices and this bound is the recovery rate c. Therefore the zero coupon yield spread must satisfy

$$\overline{y}(t,T) - y(t,T) < -\frac{\ln c}{T-t},$$

which may not be satisfied by market prices for longer times to maturity $T-t$ and high credit spreads. E.g. for a recovery rate of $c = 50\%$ and a time to maturity of $T-t = 10$ years the maximal (continuously compounded) credit spread is $h = 6.93\%$.

Despite these different properties of the two modelling approaches, with a suitable choice of (time dependent or stochastic) parameters, both models can be transformed into each other: The value of the security in default is only expressed in different numeraires, once in terms of defaultable bonds and once in terms of default-free bonds. Both approaches are therefore equivalent and one should use the specification that is best suited for the issue at hand.

3.3.5 Implied Survival Probabilities

In the equivalent recovery model it is easy to recover *implied survival probabilities* from a given term structure of defaultable bond prices and a given value for c. From equation (3.15) we have

$$\widetilde{P}(t,T) = \frac{\overline{B}_0(t,T)}{B(t,T)} = \frac{1}{1-c}\left(\frac{\overline{B}(t,T)}{B(t,T)} - c\right). \tag{3.16}$$

As before $\widetilde{P}(t,T)$ is the probability of survival from t to T under the T-forward measure (and also under the spot martingale measure for independence of credit spreads and interest rates).

This survival probability and the prices of defaultable zero coupon bonds $\overline{B}_0(t,T)$ under zero recovery are very useful to value survival contingent payoffs. For many pricing applications knowledge of $\overline{B}_0(t,T)$ is already sufficient. It is a great advantage of the equivalent recovery model that it allows to derive the value of a survival contingent payoff just from the defaultable and default-free term structures and an assumption about recovery rates c.

In the fractional recovery model it is not possible to derive the value of a zero-recovery defaultable bond just from knowledge of the recovery rate q, the defaultable bond price and the default-free bond prices unless the recovery rate is zero. Here a full specification of the dynamics of r and λ is needed.

Given independence of interest rates and the default intensity, the implied survival probability is the ratio of the zero coupon bond prices:

$$P(t,T) = \frac{\overline{B}_0(t,T)}{B(t,T)}$$

Typically the survival probability $P(t,T)$ will change over time because of two effects: First, if there was no default in $[t, t + \Delta t]$ this reduces the possible default times, information has arrived via the (non)-occurrence of the default. Secondly, additional default-relevant information could have arrived in the meantime.

For the analysis of the local default probability in some future time interval it is instructive to consider the *conditional* probability of survival. The probability of survival in $[T_1, T_2]$, given that there was no default until T_1 and given the information at time t is:

$$P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)} = \frac{\overline{B}_0(t,T_2)}{B(t,T_2)} \frac{B(t,T_1)}{\overline{B}_0(t,T_1)}.$$

This is a simple consequence of Bayes' rule. The probability of survival until T is the probability of survival until $s < T$ times the conditional probability of survival from s until T:

$$P(t,T) = P(t,s)P(t,s,T).$$

There is a close connection between forward rates and conditional survival / default probabilities.

Definition 3.3

The default-free simply compounded forward rate *over the period* $[T_1, T_2]$ *as seen from* t *is:*

$$F(t,T_1,T_2) = \frac{B(t,T_1)/B(t,T_2) - 1}{T_2 - T_1}$$

The zero-recovery defaultable simply compounded forward rate *over the period* $[T_1, T_2]$ *as seen from* t *is:*

$$\overline{F}(t,T_1,T_2) = \frac{\overline{B}_0(t,T_1)/\overline{B}_0(t,T_2) - 1}{T_2 - T_1}$$

Proposition 3.1
The conditional probability of default *over* $[T_1, T_2]$ *is given by:*

$$\frac{P^{\text{def}}(t, T_1, T_2)}{T_2 - T_1} = \frac{\overline{F}(t, T_1, T_2) - F(t, T_1, T_2)}{1 + (T_2 - T_1)\overline{F}(t, T_1, T_2)}.$$

The marginal probability of default *at time* T *is the spread of the continuously compounded defaultable forward rate over the default-free forward rate:*

$$\lim_{\Delta t \searrow 0} \frac{P^{\text{def}}(t, T, T + \Delta t)}{\Delta t} = \overline{f}(t, T) - f(t, T).$$

Proof: (dropping the t-index)

$$\begin{aligned}
P^{\text{def}}(T_1, T_2) &= 1 - P(T_1, T_2) \\
&= 1 - \frac{\overline{B}_0(T_2)B(T_1)}{B(T_2)\overline{B}_0(T_1)} \\
&= \frac{B(T_2)\overline{B}_0(T_1) - \overline{B}_0(T_2)B(T_1)}{B(T_2)\overline{B}_0(T_1)} \\
&= \frac{B(T_2)[\overline{B}_0(T_1) - \overline{B}_0(T_2)] - \overline{B}_0(T_2)[B(T_1) - B(T_2)]}{B(T_2)\overline{B}_0(T_1)} \\
&= \frac{\overline{B}_0(T_2)}{\overline{B}_0(T_1)} \frac{\overline{B}_0(T_1) - \overline{B}_0(T_2)}{\overline{B}_0(T_2)} - \frac{\overline{B}_0(T_2)}{\overline{B}_0(T_1)} \frac{B(T_1) - B(T_2)}{B(T_2)}
\end{aligned}$$

therefore

$$\frac{P^{\text{def}}(T_1, T_2)}{T_2 - T_1} = \frac{\overline{B}_0(T_2)}{\overline{B}_0(T_1)} \Big(\overline{F}(T_1, T_2) - F(T_1, T_2)\Big),$$

and from definition 3.3 follows that

$$\frac{\overline{B}_0(T_1)}{\overline{B}_0(T_2)} = 1 + (T_2 - T_1)\overline{F}(T_1, T_2).$$

The result for the marginal default probability follows directly from taking the limit.

□

The default probability over the interval $[T_1, T_2]$ equals *the length of the interval* times *the spread of the simply compounded forward rates over the interval* times *discounting with the defaultable forward rates.*

For small time intervals, the probability of default in $[T, T + \Delta t]$ is approximately *proportional* to the length of the interval with proportionality factor $(\overline{f}(t,T) - f(t,T))$.

These results highlight two points. First, there is an intimate connection between default probabilities and credit spreads. A full term structure of credit spreads contains a wealth of information about the market's perception of the likelihood of default at each point in time. The equivalent recovery model has the advantage of making this information more easily accessible than the fractional recovery model. Unfortunately, to reach this information in a practical application, a recovery rate c is needed, and the assumption of independence of defaults and default-free term structure of interest rates must be made. There is a large degree of uncertainty about recovery rates with variation between 20% and 80%. The independence assumption will have a smaller effect on the results. This assumption will be relaxed in the next sections.

The second observation is the reason why processes like Poisson or Cox processes are so well suited for credit-spread based default modelling. These processes have intensities, and the probability of jump of a point process with an intensity is approximately proportional to the length of the time interval considered (for small intervals). The proportionality factor is the intensity at that point. This property is exactly equivalent to the second equation in proposition 3.1, and it also gives a link to the model of forward credit spreads in the previous chapter. But proposition 3.1 is also valid for default models that are not based on an intensity model.

3.4 Direct Valuation of Credit Risk Derivatives

In this section we begin the pricing of credit derivatives in a framework that is independent of the specification of the model.

We first concentrate on credit derivatives that can be priced directly off the term structures of interest rates and defaultable bond prices. Among these, the default digital put option and the default digital swap will be analysed first, as these products are the simplest and the pricing of the other products can often be reduced to the pricing of default digital puts. Next are defaultable floating rate notes[12] and the default swap. Finally, intermediate valuation formulae are derived for credit spread options and options on defaultable bonds.

[12]Strictly speaking the defaultable floating rate note is not a credit derivative. It is included here because its valuation is closely linked to the valuation of the default swap.

Assumption 3.4

The following data is needed for all maturities $T > 0$:

- *the default-free term structure of bond prices*

$$B(0,T) = \mathbf{E}\left[\, e^{-\int_0^T r(s)ds} \,\right]$$

- *the defaultable bond prices*

$$\overline{B}(0,T)$$

- *the defaultable bond prices under zero recovery*

$$\overline{B}_0(0,T) = \mathbf{E}\left[\, e^{-\int_0^T r(s)+\lambda(s)ds} \,\right].$$

At some points we will assume that the term structures of interest rates and credit spreads are independent.

The assumptions about the available data are open to criticism. One of the largest difficulties in credit risk modelling is the scarcity of useful data, and only rarely (maybe for some sovereign issuers) a full term structure of defaultable bond prices is available, the same applies to survival probabilities or recovery rates. These data problems will have to be addressed separately, here we just point out that all of the defaultable inputs (recovery rates, $\overline{B}_0(0,T)$ or $P(0,T)$) can also stem from other sources than market prices, e.g. fundamental analysis, a different credit risk model or historical data of the same rating class or industry. The only requirement is that this information is given under the martingale measure.

The independence of credit spreads and interest rates will yield the survival probabilities

$$P(0,T) = \mathbf{E}\left[\, e^{-\int_0^T \lambda(s)ds} \,\right] = \overline{B}_0(0,T)/B(0,T).$$

If this assumption is dropped, we need knowledge of the full dynamics of the term structures of credit spreads, defaults and interest rates. This case will be treated in the following sections.

Alternatively to giving the defaultable bond prices under zero recovery we can also assume that an expected recovery rate is given for the equivalent recovery model. As mentioned in the previous section, it is then straightforward to imply zero-recovery defaultable bond prices from bond prices that are given under positive recovery.

3.4.1 Forms of Payment for Default Protection

There are two alternative ways in which prices for credit derivatives can be quoted: Either the price for the default protection is expressed in terms of an *up-front* fee D that is payable at $t = 0$, or a default protection fee S has to be paid *in regular intervals* until a default has happened. The latter alternative is very popular because of its similarity to swap contracts, the only difference is that the regular payments end at default. The difference is the numeraire in which the default protection is priced.

To convert the two representations we observe the following:
The value of receiving S at the times $T_i, i = 1, \dots N$ until a default is at time 0:

$$S \sum_{i=0}^{N} \overline{B}_0(0, T_i),$$

and for continuous payment (receiving sdt from T_0 until default or T_N) the value at time 0 is

$$s \int_{T_0}^{T_N} \overline{B}_0(0, t) dt. \tag{3.17}$$

These equations remain valid under correlation of r and λ.

To find the regular payments S or s that correspond to a given up-front price D we therefore only have to calculate

$$S = \frac{D}{\sum_{i=0}^{N} \overline{B}_0(0, T_i)} \qquad \text{or} \qquad s = \frac{D}{\int_{T_0}^{T_N} \overline{B}_0(0, t) dt}. \tag{3.18}$$

In the following sections we will therefore only calculate the up-front prices D of the credit derivatives.

3.4.2 Default Digital Payoffs

The default digital put option has the payoff 1 *at the time of default*. The timing is important because it determines until when the payoff has to be discounted, but we will consider a simpler specification in a first step:

3.4.2.1 Payoff at Maturity

As a first example consider an European-style default digital put which pays off 1 *at* T iff there has been a default at some time before (or including) T. Its value is easily found (τ denotes the time of the first default):

$$\begin{aligned} D =& \mathbf{E}\left[\, e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\right] \\ =& \mathbf{E}\left[\, \mathbf{E}\left[\, e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, \{\lambda(s)\}_{s\leq T} \right] \right] \\ =& \mathbf{E}\left[\, e^{-\int_0^T r(s)ds} (1 - e^{-\int_0^T \lambda(s)ds}) \,\right] \\ =& B(0,T) - \mathbf{E}\left[\, e^{-\int_0^T r(s)+\lambda(s)ds} \,\right] \\ =& B(0,T) - \overline{B}_0(0,T). \end{aligned} \tag{3.19}$$

This follows also directly from the fact that this default digital put replicates the payoff of a portfolio that is long one default-free bond and short one zero recovery defaultable bond.

3.4.2.2 Payoff at Default

In the case when the payoff takes place *at* default, the expectation to calculate is

$$D = \mathbf{E}\left[\, e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\right].$$

Conditioning on the realization of λ yields

$$D = \mathbf{E}\left[\, e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\right] = \mathbf{E}\left[\, \mathbf{E}\left[\, e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, \lambda \right] \right]. \tag{3.20}$$

Now note that the probability distribution of τ (given λ) is

$$\mathbf{P}\,[\,\tau \leq T\,] = 1 - \mathbf{P}\,[\,\tau > T\,] = 1 - \exp\{-\int_0^T \lambda(s)ds\},$$

so the density of τ is

$$\lambda(t)\exp\{-\int_0^t \lambda(s)ds\}. \tag{3.21}$$

Substituting equation (3.21) into (3.20) yields for the price of the default digital put:

$$D = \mathbf{E}\left[\, \mathbf{E}\left[\, e^{-\int_0^\tau r(s)ds} \mathbf{1}_{\{\tau<T\}} \,\middle|\, \lambda \right] \right]$$

$$= \mathbf{E}\left[\int_0^T \lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds} dt \right]$$
$$= \int_0^T \mathbf{E}\left[\lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds} \right] dt, \tag{3.22}$$

where we assume sufficient regularity to allow the interchange of expectation and integration. Under independence of r and λ the expectation in the integral in equation (3.22) can be further simplified to

$$\mathbf{E}\left[\lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds} \right] = B(0,t)P(0,t)h(0,t) = \overline{B}_0(0,t)h(0,t). \tag{3.23}$$

Here $P(0,t)$ is the survival probability, and $h(0,t)$ is the associated 'forward rate' of the spreads of the zero recovery bonds:

$$P(0,t) = \overline{B}_0(0,t)/B(0,t) = \mathbf{E}\left[e^{-\int_0^t \lambda(s)ds} \right]$$
$$h(0,t) = -\frac{\partial}{\partial T} \ln P(0,T) \bigg|_{T=t}.$$

Equation (3.23) follows from the well-known fact[13] in default-free interest rate theory that the expectation of the short rate $r(T)$ at time T under the T-forward measure is the T-forward rate $f(0,T)$:

$$\mathbf{E}\left[r(t) e^{-\int_0^t r(s)ds} \right] = B(0,t)f(0,t).$$

Here we have a mathematically equivalent situation where the notation is changed such that $P(0,t)$ are our bond prices, P_t is the forward measure, $\lambda(t)$ is the short rate at t and $h(0,t)$ is the t-forward rate. Using the survival probability $P(0,t)$ as numeraire in a change of measure it therefore follows that

$$\mathbf{E}\left[\lambda(t) e^{-\int_0^t \lambda(s)ds} \right] = P(0,t)\mathbf{E}^{P_t}\left[\lambda(t) \mid \mathcal{F}_t \right] = P(0,t)h(0,t).$$

Under the equivalent recovery model we know the zero-recovery bond prices $\overline{B}_0(t,T) = \frac{1}{1-c}(\overline{B}(t,T) - cB(t,T))$, hence under independence

$$h(t,T) = -\frac{B(t,T)}{\overline{B}(t,T) - cB(t,T)} \frac{\partial}{\partial T} \left(\frac{\overline{B}(t,T)}{B(t,T)} \right) \tag{3.24}$$
$$\overline{B}_0(t,T)h(t,T) = \frac{1}{1-c} \overline{B}(t,T)(\overline{f}(t,T) - f(t,T)) \tag{3.25}$$

[13] See e.g. Sandmann and Sondermann (1997).

hold, where the defaultable forward rate $\overline{f}(t,T)$ is defined using the defaultable bonds with positive recovery: $\overline{f}(t,T) = -\frac{\partial}{\partial T} \ln \overline{B}(t,T)$.

Putting all together we gain the following representation for the value of a default digital put with maturity T:

Proposition 3.2
Consider a default digital put with maturity T and notional value 1. All prices are as seen from time 0.

(i) If the default digital put is settled at T, its value is

$$D = B(0,T) - \overline{B}_0(0,T).$$

(ii) If the default digital put is settled at default, *its value is*

$$D = \int_0^T \mathbf{E}\left[\, \lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds} \,\right] dt. \tag{3.26}$$

(iii) Assume that r and λ are independent. Then the value of the default digital put with settlement at default is

$$D = \int_0^T \overline{B}_0(0,t) h(0,t) dt. \tag{3.27}$$

(iv) Assume that r and λ are independent, and the equivalent recovery rate is c. Then the price of a default digital put is

$$D = \frac{1}{1-c} \int_0^T \overline{B}(0,t)(\overline{f}(0,t) - f(0,t)) dt. \tag{3.28}$$

The expectation in equation (3.26) will be calculated later on.

3.4.2.3 The Default Digital Swap

For the default digital payoff, the swap specification is as follows:

> Party **A** pays 1 at default if there is a default. Party **B** pays a regular fee $s(t)dt$ until default.

If $s(t)$ can be stochastic, the fair fee is the local default intensity $\lambda(t)$. The intuitive reason is that $\lambda(t)$ is the local default probability at time t, because for small Δt

$$\mathbf{E}\left[\, \mathbf{1}_{\{\tau\in[t,t+\Delta t]\}} \;\middle|\; \mathcal{F}_t \right] \approx \lambda(t)\Delta t.$$

If the default protection is re-negotiated for every small time-interval $[t, t + dt]$, then the fee should be $\lambda(t)dt$ for this interval.

Mathematically this follows from the fact that $\int_0^t \lambda(s)\mathbf{1}_{\{s\leq\tau\}}ds$ is the predictable compensator of $\mathbf{1}_{\{\tau\leq t\}}$, thus the expected values of stochastic integrals w.r.t. either of them are equal:

$$\mathbf{E}\left[\int_0^T \beta_{0,t} d\mathbf{1}_{\{\tau\leq t\}} \;\middle|\; \mathcal{F}_t \right] = \mathbf{E}\left[\int_0^T \beta_{0,t}\lambda(t)\mathbf{1}_{\{t\leq\tau\}}dt \;\middle|\; \mathcal{F}_t \right].$$

The l.h.s. of the equation is the discounted payoff of the default-insurance side of the swap (receiving 1 at default), the r.h.s. is the premium side of the swap. The expected discounted values of both sides of the swap are equal. This result is independent of the maturity of the swap and also valid for correlation between interest rates and defaults.

The previous result required a continuous adaptation of the swap rate. Typically the swap rate s is a constant that is determined at $t = 0$ for all future times $t \leq T$. Then from equation (3.18) and proposition 3.2 follows the fair swap rate for a default digital swap:

Proposition 3.3
Assume that r and λ are independent.

(i) The fair swap rate for a default digital swap is

$$s = \frac{\int_0^T \overline{B}_0(0,t)h(0,t)dt}{\int_0^T \overline{B}_0(0,t)dt}. \tag{3.29}$$

(ii) Assume that the equivalent recovery rate is c. Then the fair rate of a default digital swap is

$$s = \frac{\int_0^T \overline{B}(0,t)(\overline{f}(t,T) - f(t,T))dt}{\int_0^T \overline{B}(0,t) - cB(0,t)dt}. \tag{3.30}$$

Remark 3.3
Compare the fair default digital swap rate to the plain vanilla fixed-for-floating swap rate

in default-free interest rate theory:

$$s_r = \frac{1 - B(0,T)}{\int_0^T B(0,t)dt} = \frac{\int_0^T B(0,t)f(0,t)dt}{\int_0^T B(0,t)dt}.$$

(The equality follows from $\frac{\partial}{\partial t}B(0,t) = -f(0,t)B(0,t)$ and integrating.) We see that the equation for the default digital swap rate has a structure that is very similar to the structure of the equation for plain fixed-for-floating swap rates. The forward rate $f(0,t)$ in the default-free case has been replaced by the credit spread $h(0,t)$ and the default-free bond prices $B(0,t)$ are replaced by their defaultable counterparts $\overline{B}_0(0,t)$. In both cases the numerator denotes the value of the floating leg of the swap, and the denominator the fixed leg of the swap, where the floating leg for the default digital swap is the default contingent payment.

The fact that the default digital swap is terminated at default introduces another difference to default-free swaps: The fixed leg has to be discounted using *defaultable* bond prices ($\overline{B}_0$, the defaultable bond prices for zero recovery). If the swap was not killed at default, the fixed side would have to be discounted with default-free rates.

3.4.3 Default Swaps

As opposed to a default digital swap which only pays a lump sum at default, a default swap covers the full loss in default of a defaultable bond. There are two possible specifications:

(a) Replacement of *the difference to par:* The payoff at the time τ of default is (using the equivalent recovery model)

$$1 - cB(\tau, T).$$

A default swap on a defaultable *coupon* bond will pay off the difference between the defaulted coupon bond price and the par value of this bond. (Thus only the principal is protected.)

(b) Replacement of *the difference to an equivalent default-free bond:* The payoff is

$$(1 - c)B(\tau, T),$$

The first variant (replacement of the difference to par) is the more common specification. Both variations (and the case of a default swap on a defaultable *coupon* bond) can be priced using the results of the previous section using portfolio arguments. For simplicity, we will first only derive the spot prices for the default insurance components the fair swap rates follow then directly from equation (3.18).

3.4.3.1 Difference to Par

Consider the portfolio consisting of

- 1 defaultable bond $\overline{B}(0,T)$
- 1 default put.

This portfolio has the following payoffs:

1. if the bond survives until T:
   1 at time T;
2. if the bond defaults before T:
   1 at the default time τ (combined payoff of defaultable bond and default put).

This payoff profile can be replicated using a portfolio consisting of

- 1 defaultable bond with zero recovery $\overline{B}_0(0,T)$
- 1 default digital put.

Thus, the price of a default put equals:

$$D^{\text{def put}} = \overline{B}_0(0,T) - \overline{B}(0,T) + D^{\text{def digital put}}.$$

The zero-recovery bond price $\overline{B}_0(0,T)$ was given by assumption, the price of the default digital put $D^{\text{def digital put}}$ was derived in the previous section.

For a default put on a defaultable *coupon* bond $\widehat{\overline{B}}(0,T)$ we only need to replace the zero coupon bonds in the example above with the respective coupon bonds. The first portfolio becomes

- 1 defaultable coupon bond $\widehat{\overline{B}}(0,T)$
- 1 default put on this bond,

and the second portfolio will be

- 1 defaultable coupon bond with zero recovery $\widehat{\overline{B}}_0(0,T)$
- 1 default digital put on the principal amount of this bond.

Again both portfolios will have identical payoff streams. The price of the zero-recovery defaultable coupon bond can be derived from the term structure of zero-recovery defaultable *zero-coupon* bonds. Note that the default digital put in the second portfolio is only on the principal of the coupon bond.

$$D^{\text{def put}} = \widehat{\overline{B}}_0(0,T) - \widehat{\overline{B}}(0,T) + D^{\text{def digital put}}.$$

3.4.3.2 Difference to default-free

If the default put pays off the price difference of the defaulted bond to an equivalent default-free bond, then the combination of this default put with a defaulted bond yields a default-free bond. Therefore its price must be the difference between the default-free bond and the defaultable bond:

$$D^{\text{def put}} = B(0,T) - \overline{B}(0,T).$$

This result is independent of correlation between credit spreads and interest rates or any assumptions about the recovery rates in default.

3.4.4 Defaultable FRNs and Default Swaps

A default swap on a defaultable floating rate note (FRN) can be used to set up a perfect hedge even for stochastic recovery rates and correlation between spreads and interest rates. We use this property to give a characterisation of the default swap rate in terms of the credit spread of defaultable FRNs.

Assume the following payoffs: The defaultable FRN pays the floating rate $r(t)$ plus a constant spread $\bar{s}$ per time. The default swap rate is s and the default swap pays the loss to par in default. The default-free FRN pays the floating rate r. Both floaters have a final payoff of 1. The payoffs are shown in table 1.

If the defaultable FRN trades at par at $t = 0$, i.e. $\overline{F}(0) = 1$ then initially (at $t = 0$), at defaults and at $t = T$ we have

defaultable FRN + default swap = default-free FRN.

As both the spread $\bar{s}$ of the defaultable FRN and the default swap rate s are constant they must coincide. Therefore

$$\bar{s} = s,$$

	defaultable FRN	default swap	default-free FRN
Price at $t = 0$	$\overline{F}(0)$	0	1
periodic payments	$(r(t) + \bar{s})dt$	$-sdt$	$r(t)dt$
final payoff at $t = T$	$1 + (r(t) + \bar{s})dt$	$-sdt$	$1 + r(t)dt$
value at default	$1 - c$	c	1

Table 3.1: The Payoffs of defaultable FRN, default swap and default-free FRN.

Assume there is a defaultable FRN that trades at par and that has a coupon spread of $\bar{s}$ *over the default-free rate* r*. Then* $s = \bar{s}$ *is the fair swap rate for a default swap of the same maturity.*

This argument only uses a simple comparison of payoff schedules, it does not use any assumptions about the dynamics and distribution of default-free interest rates or credit spreads or about the recovery rates. If the FRNs only pay coupons at discrete time interval this relationship is only approximately valid, with an exact fit at the coupon dates because only then the default-free FRN is worth exactly 1.

3.4.5 Credit Spread Forwards

A *credit spread forward* is a contract to exchange at a future time T_2 against a fixed credit spread $\bar{s}$, the credit spread that a defaultable bond $\overline{B}(T_1, T_2)$ had over an equivalent default-free bond $B(T_1, T_2)$ at time T_1. Thus the spread is fixed at T_1 and exchanged at T_2. The payoff function is

$$\frac{1/\overline{B}(T_1, T_2) - 1}{T_2 - T_1} - \frac{1/B(T_1, T_2) - 1}{T_2 - T_1} - \bar{s} \qquad \text{at } T_2. \tag{3.31}$$

The credit spread forward is the adaptation of a classical forward rate agreement (FRA) to credit spreads. Here the defaultable simply compounded interest rate is defined using defaultable bond prices with positive recovery (and not zero recovery as in definition 3.3). If a default has happened before T_1 and the defaulted bond is used for the calculation, the defaultable rate in equation (3.31) will become very large thus yielding a very high payoff to the receiver of the credit spread.

It should be noted that the classical replication portfolio for a FRA cannot be used to replicate the defaultable rate $\frac{1}{T_2-T_1}(1/\overline{B}(T_1,T_2)-1)$. The classical replication portfolio is:

- Invest \$ $B(t,T_1)$ in $B(t,T_1)$ at t (to get 1 of these bonds).
- Payoff at T_1 is \$ 1.
- Invest \$1 in $B(T_1,T_2)$ at T_1 (to get $1/B(T_1,T_2)$ of these bonds).
- Payoff at T_2 is $1/B(T_1,T_2)$.

This strategy will replicate $1/B(T_1,T_2)$ at T_2. To replicate -1 at T_2 we need to short one zero bond $B(t,T_2)$ with maturity T_2, and for the factor $1/(T_2-T_1)$ the whole strategy has to be done $1/(T_2-T_1)$ times. If this strategy is attempted with defaultable bonds $\overline{B}(t,T)$ it will fail if defaults happen before T_1.

The payoff $1/\overline{B}(T_1,T_2)$ at T_2 is unaffected by defaults in $[T_1,T_2]$, thus one needs to invest in $1/\overline{B}(T_1,T_2)$ default free bonds $B(T_1,T_2)$ at T_1 to replicate $1/\overline{B}(T_1,T_2)$ at T_2. This will cost $B(T_1,T_2)/\overline{B}(T_1,T_2)$ at T_1. The payoff of the credit spread forward is thus equivalent to

$$\frac{1}{T_2-T_1}\left(\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}-1\right)-\bar{s}B(T_1,T_2) \qquad \text{at } T_1. \tag{3.32}$$

The key term for the pricing of the credit spread forward is

$$\mathbf{E}\left[\ \beta_{T_1}\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}\ \right]=\mathbf{E}\left[\ \beta_{T_1}\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}\mathbf{1}_{\{\tau>T_1\}}+\beta_{T_1}\frac{1}{c}\mathbf{1}_{\{\tau\leq T_1\}}\ \right], \tag{3.33}$$

under the equivalent recovery model. Using the Cox process properties of the default event equation (3.33) is readily simplified to

$$\mathbf{E}\left[\ \beta_{T_1}\frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)}\ \right]=\mathbf{E}\left[\ e^{-\int_t^{T_1}r(s)+\lambda(s)ds}\frac{B(T_1,T_2)}{\widetilde{B}(T_1,T_2)}\ \right]+\frac{1}{c}(B(t,T_1)-\overline{B}_0(t,T_1)). \tag{3.34}$$

Summing up, under the equivalent recovery model the value of a credit spread forward agreement can be written as

$$D^{\mathrm{CSF}}=\frac{1}{T_2-T_1}\mathbf{E}\left[\ e^{-\int_t^{T_1}r(s)+\lambda(s)ds}\frac{B(T_1,T_2)}{\widetilde{B}(T_1,T_2)}\ \right] \tag{3.35}$$

$$+\frac{1}{T_2-T_1}\frac{1}{c^2}((1-c^2)B(t,T_1)-\overline{B}(t,T_1))-\bar{s}B(t,T_2). \tag{3.36}$$

The fair forward credit spread rate $\bar{s}$ for the credit spread forward is thus

$$\bar{s} = \frac{1}{B(t,T_2)(T_2-T_1)} \Big(\mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds} \frac{B(T_1,T_2)}{\widetilde{B}(T_1,T_2)} \right] + \frac{1}{c^2}((1-c^2)B(t,T_1) - \overline{B}(t,T_1)) \Big).$$

Under the fractional recovery model the key term (3.33) for the pricing of the credit spread forward resolves to

$$\begin{aligned} \mathbf{E}\left[\beta_{T_1} \frac{B(T_1,T_2)}{\overline{B}(T_1,T_2)} \right] &= \mathbf{E}\left[\beta_{T_1} \frac{B(T_1,T_2)}{Q(T_1)\widetilde{B}(T_1,T_2)} \right] \\ &= \mathbf{E}\left[e^{\int_0^{T_1} \frac{q}{1-q}\lambda(s)-r(s)ds} \frac{B(T_1,T_2)}{\widetilde{B}(T_1,T_2)} \right]. \end{aligned} \tag{3.37}$$

In the step to equation (3.37) we used the following derivation of the expectation of $Q(T_1)^{-1}$, which is also an application of the Cox process properties of N for multiple jumps:

$$\begin{aligned} \mathbf{E}\left[\frac{1}{Q(T)} \,\middle|\, \lambda(t)\, t \geq 0 \right] &= \sum_{n=1}^{\infty} (1-q)^{-n} \mathbf{P}\left[\, N(T) = n \mid \lambda(t)\, t \geq 0 \,\right] \\ &= \sum_{n=1}^{\infty} (1-q)^{-n} \frac{1}{n!} \left(\int_0^T \lambda(s)ds \right)^n e^{-\int_0^T \lambda(s)ds} \\ &= e^{-\int_0^T \lambda(s)ds} \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{1}{1-q} \int_0^T \lambda(s)ds \right)^n \\ &= e^{\frac{q}{1-q} \int_0^T \lambda(s)ds}. \end{aligned}$$

3.4.6 Credit Spread Put Options

Definition 3.4
A Credit Spread Put *on a defaultable bond $\overline{B}(t,T_2)$ with maturity $T_1 < T_2$ and strike spread $\bar{s}$ gives the holder the right to sell the defaultable bond at time T_1 at a price that corresponds to a yield spread of $\bar{s}$ above the yield of an (otherwise identical) default-free bond $B(T_1,T_2)$.*

Define the exchange ratio $\bar{S} := e^{-\bar{s}(T_2-T_1)}$. Then the credit spread put entitles the holder to exchange *one defaultable bond $\overline{B}(T_1,T_2)$ for $\bar{S}$ default-free bonds $B(T_1,T_2)$ at time T_1. The payoff function is:*

$$(\bar{S}B(T_1,T_2) - \overline{B}(T_1,T_2))^+. \tag{3.38}$$

Depending on the specification the option can either survive a default or be knocked out by it. In the first case the payoff is conditioned on the full defaultable bond price $\overline{B}(T_1, T_2)$, and the default risk is borne by the writer of the security. The holder of the security is protected by the contract against any losses worse than a final credit spread of $\bar{s}$ and against all defaults but not against interest rate risk. The exchange option is therefore well suited to lay off most of the credit exposure in the underlying instrument while keeping some limited exposure to the credit spread of the underlying. We assume that the strike spread $\bar{s}$ is small enough (or the loss in default large enough) to ensure exercise of the option after a default.

Lemma 3.4

The price of the credit spread put is given by the following expectation:

$$\begin{aligned} D_t = &\mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds} B(T_1,T_2)(\bar{S} - \widetilde{P}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\ &+ \bar{S}B(t,T_2) - \overline{B}(t,T_2) \\ &- \mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds} B(T_1,T_2)(\bar{S} - \widetilde{P}(T_1,T_2)) \;\middle|\; \mathcal{F}_t \right]. \end{aligned} \tag{3.39}$$

The price of the credit spread put with knockout at default is given by

$$D_t = \mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds} B(T_1,T_2)(\bar{S} - \widetilde{P}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right]. \tag{3.40}$$

Although equations (3.39) and (3.40) do not represent a closed-form solution yet, they have several uses: Firstly, the problem is now accessible to the standard techniques of interest-rate theory. In the next section we will apply these techniques to derive full closed form solutions. Secondly, a numerical solution of equations (3.39) and (3.40) is feasible, again with standard techniques. In a Monte-Carlo simulation the elimination of the indicator functions $\mathbf{1}_{\{\tau>T_1\}}$ will speed up convergence significantly[14]. Finally, the analytical techniques can be also used as a pre-processing step for more complex problems (e.g. coupon bond options), before a numerical solution is attempted.

The credit spread put option price in equation (3.39) can be decomposed in two parts: A spread protection part (the first line) and a default protection part (the second and third lines). If the option is knocked out at default only the spread protection part remains.

[14]The speed of convergence without default indicators is about 100 times faster than direct simulation.

Proof: The expectation to evaluate is (assuming no defaults so far $Q(t) = 1$)

$$\begin{aligned}
D_t =&\mathbf{E}\left[\, \beta_{t,T_1}(\bar{S}B(T_1,T_2) - \overline{B}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \,\right] \\
=&\mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - \widetilde{B}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\
&+ \mathbf{E}\left[\, \mathbf{1}_{\{\tau\le T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - Q(T_1)\widetilde{B}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right],
\end{aligned}$$

where we split up the payoff in the case without intermediate default ($\mathbf{1}_{\{\tau>T_1\}}$) and the case with intermediate default ($\mathbf{1}_{\{\tau\le T_1\}}$). If there has been an intermediate default, we assumed that the option will be exercised and we omit the maximum sign for that event:

$$\begin{aligned}
=&\mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - \widetilde{B}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\
&+ \mathbf{E}\left[\, \mathbf{1}_{\{\tau\le T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - Q(T_1)\widetilde{B}(T_1,T_2)) \;\middle|\; \mathcal{F}_t \right],
\end{aligned}$$

and then substitute $\mathbf{1}_{\{\tau\le T_1\}} = 1 - \mathbf{1}_{\{\tau>T_1\}}$ to reach

$$\begin{aligned}
=&\mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - \widetilde{B}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\
&+ \mathbf{E}\left[\, \beta_{t,T_1}(B(T_1,T_2)\bar{S} - Q(T_1)\widetilde{B}(T_1,T_2)) \;\middle|\; \mathcal{F}_t \right] \\
&- \mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - Q(T_1)\widetilde{B}(T_1,T_2)) \;\middle|\; \mathcal{F}_t \right] \\
=&\mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - \widetilde{B}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\
&+ B(t,T_2)\bar{S} - \overline{B}(T_1,T_2) \\
&- \mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}(B(T_1,T_2)\bar{S} - \widetilde{B}(T_1,T_2)) \;\middle|\; \mathcal{F}_t \right].
\end{aligned}$$

In the last line we used the fact that $\mathbf{1}_{\{\tau>T_1\}}Q(T_1) = \mathbf{1}_{\{\tau>T_1\}}$, the indicator function of 'no default until T_1' switches off the 'influences $Q(T_1)$ of default until T_1'.

Next we have to eliminate the indicator functions from the expectations. This is done by conditioning on the realisation of λ and using the Cox process properties of the default triggering process. Using $\widetilde{B}(T_1,T_2) = B(T_1,T_2)\widetilde{P}(T_1,T_2)$ this yields:

$$\begin{aligned}
&\mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}B(T_1,T_2)(\bar{S} - \widetilde{P}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\
=&\mathbf{E}\left[\, \mathbf{E}\left[\, \mathbf{1}_{\{\tau>T_1\}}\beta_{t,T_1}B(T_1,T_2)(\bar{S} - \widetilde{P}(T_1,T_2))^+ \;\middle|\; \lambda \right] \;\middle|\; \mathcal{F}_t \right] \\
=&\mathbf{E}\left[\, e^{-\int_t^{T_1}\lambda(s)ds}\beta_{t,T_1}B(T_1,T_2)(\bar{S} - \widetilde{P}(T_1,T_2))^+ \;\middle|\; \mathcal{F}_t \right].
\end{aligned}$$

The second expectation is treated similarly to reach equation (3.39).

□

3.4.7 Put Options on Defaultable Bonds

The credit risk derivatives considered so far conditioned on the *spread* of the defaultable bond over a default-free bond, but the reference to a default-free bond is not necessary, we can also consider derivatives on the defaultable bond alone, e.g. a Put or call on a defaultable bond. Again we consider the cases when the Put covers default losses, and when it is killed at default.

Definition 3.5
A European put with maturity T_1 and strike K on a defaultable bond $\overline{B}(t, T_2)$ pays off

$$(K - \overline{B}(T_1, T_2))^+. \tag{3.41}$$

If the put is knocked out at default the payoff is

$$\mathbf{1}_{\{\tau > T_1\}}(K - \overline{B}(T_1, T_2))^+. \tag{3.42}$$

This derivative protects the buyer against all risks: Interest rates, credit spreads or defaults that could move the defaultable bond price below K. Assuming exercise of the option in default we reach:

Lemma 3.5
The price of the Put option on a defaultable bond is

$$\begin{aligned} D_t =& \mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds}(K - \widetilde{B}(T_1, T_2))^+ \;\middle|\; \mathcal{F}_t \right] \\ &+ B(t, T_1)K - \overline{B}(t, T_2) \\ &- \mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds}(K - \widetilde{B}(T_1, T_2)) \;\middle|\; \mathcal{F}_t \right]. \end{aligned} \tag{3.43}$$

If the option is knocked out at default its price is

$$D_t = \mathbf{E}\left[e^{-\int_t^{T_1} r(s)+\lambda(s)ds}(K - \widetilde{B}(T_1, T_2))^+ \;\middle|\; \mathcal{F}_t \right]. \tag{3.44}$$

Proof: Replace $\bar{S}B(T_1, T_2)$ with $KB(T_1, T_1) = K$ in lemma 3.4. Then the proof carries through exactly as the proof to lemma 3.4.

□

Like the credit spread put option the plain put option can be decomposed in a default protection component and a price protection component. If the option is knocked out at default it has only the value of the price protection component.

It is important to note that the value of this option is *not* the value of an European Put on a bond in a world where the short rate is $\overline{r} = r + q\lambda$. (One might be lead to this view by the simple representation (3.10) of the defaultable bond prices.) This view would ignore the large jumps the bond price makes at defaults and only price the spread component.

To illustrate this point, we introduce a new security with the following payoff function under the fractional recovery model:

$$Q(T_1)(K - \widetilde{B}(T_1, T_2))^+. \tag{3.45}$$

At each default the face value of the defaultable bond is reduced by a factor $1 - q$. This payoff function is also reduced by the same factor, such that this option always covers exactly one defaultable bond, the size of the protection is adjusted to the 'size' (=face value) of the security. Again very similarly to (3.39) we reach the expectation

$$\mathbf{E}\left[e^{-\int_t^{T_1} r(s)+q\lambda(s)ds}(K - \widetilde{B}(T_1, T_2))^+ \;\middle|\; \mathcal{F}_t \right],$$

which is exactly identical to the expectation that has to be evaluated to price a put option on a zero coupon bond in a world where $\overline{r} = r + q\lambda$ is the riskless short rate. Unfortunately the specification of this payoff function heavily uses the fractional recovery model as input, specifically it conditions on loss quota (Q) and loss-quota adjusted price $\widetilde{B}$ separately. In real-world applications it would be impossible to separate these two quantities in a meaningful way, therefore the payoff function (3.45) remains an academic example.

3.5 Models

In the previous section we could derive some pricing formulae for credit derivatives without having to specify the dynamics we assume for credit spreads and interest rates. This was done by assuming independence of the realisations of r and λ.

For the pricing of credit derivatives with option features and to allow for correlation between r and λ we need a full specification of the dynamics of the default-free interest rate r and the intensity of the default process λ. A suitable specification should have the following properties:

- Both r and λ are stochastic. Stochastic default-free interest rates are indispensable for fixed-income analysis, and a stochastic default intensity is required to reach stochastic credit spreads which is necessary for meaningful prices for credit spread options.
- The dynamics of r and λ are rich enough to allow for a realistic description of the real-world prices. Duffie and Singleton (1997) and Duffee (1995) come to the conclusion that in many cases a multifactor model for the credit spreads is necessary.
- There should be scope to include correlation between credit spreads and default-free interest rates.
- It is desirable to have interest rates and credit spreads that remain positive at all times. Although negative credit spreads or interest rates represent an arbitrage opportunity, relaxing this requirement in favour of a Gaussian specification is still acceptable because of the analytical tractability that is gained. The Gaussian specification should then be viewed as a local approximation to the real-word dynamics rather than as a fully closed model. Furthermore, many important effects are more easily understood in the Gaussian setup.

Therefore we chose two alternative setups:

1. A *multifactor Gaussian* setup. Here there is the possibility of reaching negative credit spreads and interest rates with positive probability, but a high degree of analytical tractability is retained and the full term structures of bond prices and volatilities can be specified.
2. A *multifactor Cox Ingersoll Ross (CIR) (1985b)* setup, following mainly Jamshidian (1996).[15] This model setup gives us the required properties while still retaining a large degree of analytical tractability. Furthermore, models of credit spreads of the CIR square-root type have been estimated by Duffie and Singleton (1997) and Duffee (1995).

Both specifications use the fractional recovery model, but most the results can be transferred to the equivalent recovery model: By assuming full loss in default $q = 1$, the specification can be viewed as a specification of the dynamics of zero-recovery defaultable bond prices $\overline{B}_0(t,T)$. Combining these dynamics of the zero-recovery defaultable

[15]Related models can also be found in Jamshidian (1987; 1995), Duffie and Kan (1996), Chen and Scott (1995) and Longstaff and Schwartz (1992).

bond prices with an assumption on the equivalent recovery rate c will yield a specification in terms of the equivalent recovery model where defaultable bond prices are given by $\overline{B}(t,T) = cB(t,T) + (1-c)\overline{B}(t,T)$.

3.6 The Multifactor Gaussian Model

In this section we present the setup of the Gaussian specification of the fractional recovery model. This specification is problematic as it allows credit spreads and interest rates to become negative with positive probability. This problem is well known from default-free Gaussian interest-rate models and it is partly compensated by the analytical tractability gained, provided the probability of these events remains low. This setup should therefore be viewed as an approximation to more realistic models of credit spread and interest rates, and despite this drawback it will yield some valuable insights.

Assumption 3.5
The following asset price dynamics under the martingale measure are given:

- *default-free bond prices* $B(t,T)$

$$\frac{dB(t,T)}{B(t,T)} = r(t)dt + a(t,T)dW,$$

- *and defaultable bond prices* $\overline{B}(t,T) = Q(t)\widetilde{B}(t,T)$

$$\begin{aligned} dQ(t) &= -Q(t-)q\,dN(t) \\ \frac{d\widetilde{B}(t,T)}{\widetilde{B}(t,T)} &= (r(t) + q\lambda(t))dt + \overline{a}(t,T)dW. \end{aligned}$$

The Brownian motion dW is d-dimensional, the volatilities $a(t,T)$ and $\overline{a}(t,T)$ are d-dimensional deterministic functions of time t and time to maturity T only, and at $t = 0$ there has been no default: $\tau > 0$ and $Q(0) = 1$ and $\widetilde{B}(0,T) = \overline{B}(0,T)$.

This specification is equivalent to a Gaussian HJM model with the following forward rate volatilities for the bond prices:

$$\begin{aligned} \sigma(t,T) &= -\frac{\partial}{\partial T}a(t,T) \qquad \overline{\sigma}(t,T) = -\frac{\partial}{\partial T}\overline{a}(t,T) \\ \sigma^h(t,T) &= -\frac{\partial}{\partial T}(\overline{a}(t,T) - a(t,T)). \end{aligned}$$

This setup can be used to model any degree of correlation and de-correlation that is desired. Multiplication of the volatilities is defined by the scalar product in $\mathbb{R}^d$: $a\overline{a} = \sum_{i=1}^{d} a_i \overline{a}_i$.

3.7 Credit Derivatives in the Gaussian Model

3.7.1 Implied Survival Probabilities

In the Gaussian setup we can derive the implied survival probability from t to T in closed form. Given $\tau > t$ the survival probability is defined as:

$$P(t,T) = \mathbf{E}\left[\, e^{-\int_t^T \lambda(s)ds} \;\middle|\; \mathcal{F}_t \right].$$

Lemma 3.6
The dynamics of the survival probabilities are

$$\frac{dP(t,T)}{P(t,T)} = \lambda(t)dt + a^p(t,T)dW \tag{3.46}$$

$$a^p(t,T) = \frac{1}{q}\left(\overline{a}(t,T) - a(t,T)\right), \tag{3.47}$$

and the initial values $P(0,T)$ are given by:

$$P(0,T) = \left(\frac{\overline{B}(0,T)}{B(0,T)}\right)^{\frac{1}{q}} \exp\left\{-\frac{1}{2q^2}\int_0^T (\overline{a}(s,T) - a(s,T))[(1+q)a(s,T) - (1-q)\overline{a}(s,T))]ds\right\}. \tag{3.48}$$

or

$$P(0,T) = \left(\frac{\overline{B}(0,T)}{B(0,T)}\right)^{\frac{1}{q}} \exp\left\{-\frac{1}{2}\int_0^T a^p(s,T)[a(s,T) + \overline{a}(s,T) - a^p(s,T)]ds\right\}. \tag{3.49}$$

Proof: See appendix A.1 on page 113.

□

For zero recovery (full loss $q = 1$ in default) equation (3.48) reduces to

$$P(0,T) = \frac{\overline{B}(0,T)}{B(0,T)} \exp\{-\int_0^T a^p(s,T)a(s,T)ds\}. \tag{3.50}$$

Under independence of credit spreads and interest rates the implied survival probability would further reduce to $\frac{\overline{B}(0,T)}{B(0,T)}$ as it was given in section 3.3.5. Hence the factor

$$e^{-\int_0^T a^p(s,T)a(s,T)ds}$$

represents the influence of correlation between spreads and interest rates on the implied default probabilities. There is an intuitive explanation of the direction of the effect:

If interest rates and credit spreads are positively correlated ($a^p a > 0$) this means that defaults are slightly more likely in states of nature when interest rates are high. Because of the higher interest rates these states are discounted more strongly when they enter the price of the defaultable bond, and conversely states with low interest rates enter with less discounting and simultaneously fewer defaults. To reach a *given* price for a defaultable bond, the absolute default likelihood must therefore be higher. This implies a lower survival probability which is also what equation (3.50) yields for $a^p a > 0$. The argument runs conversely for negative correlation $a^p a < 0$.

3.7.2 The Survival Contingent Measure

The following lemma describes the change of measure that is necessary to value survival contingent payoffs in this framework.

Lemma 3.7
Let $X \geq 0$ be a $\mathcal{F}_T$-measurable random variable and $\tau > t$. In the Gaussian model framework, the time t value of receiving X at T given no default has happened before T *is*

$$\mathbf{E}^Q\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}}X \;\middle|\; \mathcal{F}_t \right] = \overline{B}_0(t,T)\mathbf{E}^{P_S}\left[\, X \mid \mathcal{F}_t \,\right], \tag{3.51}$$

where

$$\overline{B}_0(t,T) = B(t,T)P(t,T)\exp\{\int_t^T a^p(s,T)a(s,T)ds\} \tag{3.52}$$

$$= \left(\frac{\overline{B}(t,T)}{B(t,T)^{1-q}}\right)^{\frac{1}{q}} \exp\left\{\frac{1-q}{2q}\int_t^T (\overline{a}(s,T) - a(s,T))^2 ds\right\}. \tag{3.53}$$

The measure P_S is called the survival contingent measure. *It is defined by the Radon-Nikodym density*

$$dP_{ST} = M_T dQ_T \tag{3.54}$$

where $dP_{ST} = dP_S|\, \mathcal{F}_T$ and $dQ_T = dQ|\, \mathcal{F}_T$, and

$$M_u = \mathcal{E}\left(\int_t^u (a^p(s,T) + a(s,T))dW_s\right), \qquad u \geq t. \tag{3.55}$$

The Q-Brownian motion W_s is transformed into a P_S-Brownian motion via

$$dW_s = dW_s^{P_S} + (a^p(s,T) + a(s,T))ds. \tag{3.56}$$

Proof: See appendix A.2 on page 115.

□

If the survival contingent measure is known, the valuation of the payoff X can be decoupled from the default valuation which is represented by $\overline{B}_0(t,T)$. The numeraire used in this change of measure is

$$\mathbf{E}\left[\, e^{-\int_t^T r(s)+\lambda(s)ds} \;\middle|\; \mathcal{F}_t \right]$$

which is almost exactly the price of a zero-recovery defaultable bond:

$$\overline{B}_0(t,T) = \mathbf{1}_{\{\tau > t\}} \mathbf{E}\left[\, e^{-\int_t^T r(s)+\lambda(s)ds} \;\middle|\; \mathcal{F}_t \right].$$

For no previous defaults ($\tau > t$) both expressions coincide. In equations (3.52) and (3.53) the lemma also gives the value of a zero-recovery defaultable bond with maturity T in the fractional recovery setup.

Using lemma 3.6 we can now derive the values of some credit derivatives that were introduced in section 3.2.

3.7.3 Default Digital Payoffs

The price for a default digital put is according to equation (3.22)

$$D = \int_0^T \mathbf{E}\left[\, \lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds} \,\right] dt.$$

The resulting price for the default digital put is:

Proposition 3.8
The price of a default digital put with maturity T and payoff 1 at default is in the multi-factor Gaussian model framework

$$D = \int_0^T P(0,t)B(0,t)e^{\int_0^t a(s,t)a^p(s,t)ds} \left[\lambda(0,t) + \int_0^t a(s,t)\sigma^p(s,t)ds\right] dt \tag{3.57}$$

where $\lambda(0,t) := -\frac{\partial}{\partial t}\ln P(0,t)$ and $\sigma^p(s,t) := -\frac{\partial}{\partial t}a^p(s,t)$ are given in lemma 3.6.

Proof: See appendix A.3 on page 115.

□

3.7.4 The Credit Spread Put

Proposition 3.9
The price of the Credit Spread Component of the Credit Spread Put is given by:

$$\begin{aligned} C^{CSP} =& N(d_1)\, \bar{S}\, B(t,T_2)\, P(t,T_1) \exp\Big\{\int_t^{T_1} a(s,T_2)a^p(s,T_1)ds\Big\} \\ &- N(d_2)\, \overline{B}(t,T_2)\, \big(P(t,T_1)\big)^{1-q} \exp\{\int_t^{T_1} (1-q)a^p(s,T_1)(\overline{a}(s,T_2) - \frac{1}{2}qa^p(s,T_1))ds\} \end{aligned}$$

where

$$d_1 = \frac{K + F + \frac{1}{2}V}{\sqrt{V}} \qquad \text{and} \qquad d_2 = d_1 - \sqrt{V}.$$

The other parameters are:

$$\begin{aligned} K &= \ln \bar{S} + \ln \frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln \frac{\overline{B}(t,T_2)}{B(t,T_2)} \\ F &= \int_t^{T_1} qa^p(s,T_1)\big[a(s,T_2) - a(s,T_1) - (1-q)(a^p(t,T_2) - a^p(t,T_1))\big]\, ds \\ V &= \int_t^{T_1} q^2(a^p(s,T_2) - a^p(s,T_1))^2\, ds. \end{aligned}$$

Proof: See appendix A.4 on page 117.

□

The parameters of this pricing formula have a number of similarities to classical Black-Scholes option prices.
The parameter K is similar to the logarithm of *'share/strike'* as it is found in the exchange option. Here we have the ratio of 'forward spread' and strike spread instead but the function is the same.
The parameter V gives the 'volatility' of the payoff. Here it is the volatility of the forward credit spread over the interval $[T_1, T_2]$.
F is a correlation parameter. Under zero recovery ($q = 1$) it reduces to zero if the volatilities of spreads a^p and default-free bond prices a generate independent dynamics.
Finally, the form of the representation of the points d_1, d_2, at which the normal distribution is evaluated, are similar to the classical Black-Scholes parameters.

The similarities to the classical Black-Scholes formula are not surprising if one bears in mind that – given that no default happens – under the multifactor Gaussian setup defaultable and default free bond prices are lognormally distributed.

For completeness we also give the price of the default-protection component of the credit spread put:

Proposition 3.10
The price of the default protection component of the credit spread put is given by

$$C^{\text{def}} = \bar{S}\; B(t,T_2)(1 - \; P(t,T_1)e^H) - \overline{B}(t,T_2)(1 - \left(P(t,T_1)e^H\right)^{1-q})$$

where

$$e^H = \exp\{\int_t^{T_1} a(s,T_2)a^p(s,T_1)ds\}.$$

The default protection component of the credit spread put gives the holder the right to put a *defaulted* bond $\overline{B}(t,T_2)$ to the seller of the option at the price of $\bar{S}$ default-free bonds $B(t,T_2)$. If $\bar{S}$ is close to 1 this protection can be approximated by a default swap.

3.7.5 The Put on a Defaultable Bond

Proposition 3.11
The price of a European put option with strike $\bar{S}$, expiry T_1 and knockout at default on a defaultable bond $\overline{B}(t,T_2)$ is (under the fractional recovery model)

$$
\begin{aligned}
C^{put} &= \bar{S}\overline{B}_0(t,T_1)N(d_1) \\
&\quad - \overline{B}(t,T_2)P(t,T_1)^{1-q}\exp\{(1-q)\int_t^{T_1} a^p(s,T_1)(\overline{a}(s,T_2) - \frac{1}{2}qa^p(s,T_1))ds\}N(d_2)
\end{aligned}
$$

where

$$
\begin{aligned}
d_1 &= \frac{K + F_1 + \frac{1}{2}V}{\sqrt{V}} \qquad d_2 = \frac{K + F_2 - \frac{1}{2}V}{\sqrt{V}} \\
K &= \ln\frac{\bar{S}\overline{B}(t,T_1)}{\overline{B}(t,T_2)} \\
F_1 &= -\int_t^{T_1}(\overline{a}(s,T_2) + a(s,T_1) + a^p(s,T_1))(\overline{a}(s,T_2) - \overline{a}(s,T_1))ds \\
F_2 &= F_1 + q\int_t^{T_1}(\overline{a}(s,T_2) - \overline{a}(s,T_1))a^p(s,T_1)ds \\
V &= \int_t^{T_1}(\overline{a}(s,T_2) - \overline{a}(s,T_1))^2 ds.
\end{aligned}
$$

Proof: See appendix A.5 on page 121.

□

Under the equivalent recovery model the put option on a defaultable bond is equivalent to a put option on a *portfolio* consisting of c default-free bonds $B(t,T_2)$ and $(1-c)$ defaultable bonds with zero recovery $\overline{B}_0(t,T_2)$. In this case the valuation of the option will become much more difficult and either approximations or numerical methods will have to be employed.

3.8 The Multifactor CIR Model

The multifactor CIR model is set up as follows:

Assumption 3.6
Interest rates and default intensities are driven by n independent factors x_i, $i = 1,\dots,n$

with dynamics of the CIR square-root type:

$$dx_i = (\alpha_i - \beta_i x_i)dt + \sigma_i \sqrt{x_i} dW_i. \tag{3.58}$$

The coefficients satisfy $\alpha_i > \frac{1}{2}\sigma_i^2$ *to ensure strict positivity of the factors.*

The default-free short rate r *and the default intensity* λ *are positive linear combinations of the factors* x_i *with weights* $w_i \geq 0$ *and* $\overline{w}_i$, $(0 \leq i \leq n)$ *respectively:*

$$r(t) = \sum_{i=1}^{n} w_i x_i(t). \tag{3.59}$$

$$\lambda(t) = \sum_{i=1}^{n} \overline{w}_i x_i(t). \tag{3.60}$$

With all coefficients $w_i \geq 0$ and $\overline{w}_i \geq 0$ nonnegative we have ensured that $r > 0$ and $\lambda > 0$ almost surely. Unfortunately this specification can only generate *positive* correlation between r and λ. If negative correlation is needed one could define modified factors x_i' that are negatively correlated to the x_i by $dx_i' = (\alpha_i - \beta_i x_i')dt - \sigma_i \sqrt{x_i'} dW_i$ (note the minus in front of the Brownian motion). This would complicate the analysis. Alternatively one could restrict the specification to a squared Gaussian model.

Typically only the first $m < n$ factors would describe the default-free term structure (i.e. $w_i = 0$ for $i > m$), and the full set of state variables would be used to describe the spreads. The additional $n-m$ factors for the spreads ensure that the dynamics of the credit spreads have components that are independent of the default-free interest rate dynamics. The simplest example would be independence between r and λ, where $r = x_1$ and $\lambda = x_2$: one factor driving each rate,

$$\begin{aligned} n &= 2 & m &= 1 \\ w_1 &= 1 & w_2 &= 0 \\ \overline{w}_1 &= 0 & \overline{w}_2 &= 1. \end{aligned}$$

3.8.1 Bond Prices

For a linear multiple cx_i ($c > 0$ is a positive constant) of the factor x_i the following equation gives the corresponding 'bond price' (see CIR (1985b)):

$$\mathbf{E}\left[\exp\{-\int_t^T cx_i(s)ds\} \;\middle|\; \mathcal{F}_t \right] = H_{1i}(T-t,c)e^{-H_{2i}(T-t,c)cx_i} \tag{3.61}$$

where

$$H_{1i}(T-t,c) = \left[\frac{2\gamma_i e^{\frac{1}{2}(\gamma_i+\beta_i)(T-t)}}{(\gamma_i+\beta_i)(e^{\gamma_i(T-t)}-1)+2\gamma_i} \right]^{2\alpha_i/\sigma_i^2} \tag{3.62}$$

$$H_{2i}(T-t,c) = \frac{2(e^{\gamma_i(T-t)}-1)}{(\gamma_i+\beta_i)(e^{\gamma_i(T-t)}-1)+2\gamma_i} \tag{3.63}$$

$$\gamma_i = \sqrt{\beta_i^2 + 2c\sigma_i^2}. \tag{3.64}$$

The default-free bond prices are given by

$$\begin{aligned} B(t,T) &= \mathbf{E}\left[e^{-\int_t^T r(s)ds} \;\middle|\; \mathcal{F}_t \right] = \mathbf{E}\left[e^{-\sum_i \int_t^T w_i x_i(s)ds} \;\middle|\; \mathcal{F}_t \right] \\ &= \mathbf{E}\left[\prod_i e^{-\int_t^T w_i x_i(s)ds} \;\middle|\; \mathcal{F}_t \right] = \prod_i \mathbf{E}\left[e^{-\int_t^T w_i x_i(s)ds} \;\middle|\; \mathcal{F}_t \right] \end{aligned}$$

because the factors are independent. The bond price is thus a product of one-factor bond prices

$$B(t,T) = \prod_{i=1}^n H_{1i}(T-t,w_i)e^{-H_{2i}(T-t,w_i)w_i x_i(t)}. \tag{3.65}$$

Similarly, the defaultable bond prices under fractional recovery are given by:

$$\overline{B}(t,T) = Q(t)\prod_{i=1}^n H_{1i}(T-t,w_i+q\overline{w}_i)e^{-H_{2i}(T-t,w_i+q\overline{w}_i)(w_i+q\overline{w}_i)x_i}, \tag{3.66}$$

because of $\overline{B}(t,T) = Q(t)\mathbf{E}\left[\exp\{-\int_t^T r(s) + q\lambda(s)ds\} \;\middle|\; \mathcal{F}_t \right]$, zero recovery bond prices can be derived from (3.66) by setting $q = 1$.

3.8.2 Affine Combinations of Independent Non-Central Chi-Squared Distributed Random Variables

The mathematical tools for the analysis of the model have been provided by Jamshidian (1996), who used them to price interest-rate derivatives in a default-free interest rate environment. Of these tools we need the expressions for the evaluation of the expectations that will arise in the pricing equations, and the methodology of a change of measure to remove discount factors from these expectations.

First the distribution function of an affine combination of noncentral chi-squared random variables is presented. Most of the following expressions are based upon this distribution function. Then we consider the expressions for the expectations of noncentral chi-squared random variables, and finally the change-of-measure technique that has to be applied in this context.

Definition 3.6
Let z_i, $1 \le i \le n$ be n independent, noncentral chi-square distributed random variables with ν_i degrees of freedom and noncentrality parameter[16] $\tilde{\lambda}_i$.

Let Y be an affine combination of the random variables z_i, $1 \le i \le n$ with weights η_i and offset ϵ:

$$Y = \epsilon + \sum_{i=1}^{n} \eta_i z_i. \tag{3.67}$$

We call Y a Affine combination of Noncentral Chi-squared random variables (ANC) *and denote its distribution function with χ_n^2:*

$$\mathbf{P}[\, Y \le y \,] =: \chi_n^2(\, y \,;\, \nu,\, \tilde{\lambda},\, \eta,\, \epsilon),$$

where η, ν and $\tilde{\lambda}$ are vectors giving weight, degrees of freedom and noncentrality of the z_i, and for $\epsilon = 0$ we will omit the last argument[17].

The evaluation of this distribution function can be efficiently implemented via a fast Fourier transform of its characteristic function (see e.g. Chen and Scott (1992)). Alternatively, Jamshidian gives the following formula (which is basically the transform integral of the

[16] The noncentrality parameter $\tilde{\lambda}$ is not to be confused with the intensity λ of the defaults.
[17] This is justified by equation (3.71).

characteristic function for $\eta > -\frac{1}{2}$):

$$\chi^2(y;\ \nu, \tilde{\lambda}, \eta, 0) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Psi(\frac{\nu}{2}, \tilde{\lambda}, 2\xi^2\eta^2)\ \sin(\xi y - \Phi(\nu, \tilde{\lambda}, \xi\eta))\frac{d\xi}{\xi}, \tag{3.68}$$

where $\Psi(\nu, \tilde{\lambda}, \eta) = \mathbf{E}\left[\ e^{-Y}\ \right]$ will be given in equation (3.72) and

$$\Phi(\nu, \tilde{\lambda}, \gamma) = \sum_{i=1}^{n}\left(\frac{\nu_i}{2}\arctan(2\gamma_i) + \frac{\gamma_i\tilde{\lambda}_i}{1+4\gamma_i^2}\right).$$

The integral in (3.68) is numerically well-behaved in that the limit of the integrand as $\xi \to 0$ is finite and the integrand is absolutely integrable. Using equation (3.71) this can be extended for the case of $\epsilon \neq 0$.

One advantage of this setup is that – although we are working with a n-factor model – most valuation problems can be reduced to the evaluation of this one-dimensional integral. The evaluation of the noncentral chi-squared distribution function which one encounters in the one-factor case, also requires a numerical approximation, the implementation effort of the multifactor model therefore does not seem to be very much higher than the effort required for a model with one independent factor each for interest rates and spreads.

The following lemma by Jamshidian (1996) provides most of the expressions we need:

Lemma 3.12
Let Y and Y' be ANC distributed with the same z_i, but Y with weights η and offset ϵ, and Y' ANC with weights η' and offset ϵ'.

$$Y \sim \chi^2_n(\ \cdot\ ;\ \nu,\ \tilde{\lambda},\ \eta,\ \epsilon), \qquad\qquad Y' \sim \chi^2_n(\ \cdot\ ;\ \nu,\ \tilde{\lambda},\ \eta',\ \epsilon'). \tag{3.69}$$

Let y be a constant. Then
the expectation of Y is:

$$\mathbf{E}[\ Y\ ] = \epsilon + \sum_{i=1}^{n}\eta_i(\tilde{\lambda}_i + \nu_i), \tag{3.70}$$

the distribution of Y is:

$$\mathbf{P}[\ Y \leq y\ ] = \chi^2_n(y - \epsilon;\ \nu, \tilde{\lambda}, \eta), \tag{3.71}$$

the expectation of e^{-Y} *is:*

$$\mathbf{E}\left[\ e^{-Y}\ \right] = e^{-\epsilon} \prod_{i=1}^{n} \frac{1}{(1+2\eta_i)^{\nu_i/2}} \exp\left\{ -\frac{\eta_i \tilde{\lambda}_i}{1+2\eta_i} \right\} \tag{3.72}$$

the value of a call option on e^{-Y} *with strike* e^{-y} *is:*

$$\begin{aligned}\mathbf{E}\left[\ (e^{-Y} - e^{-y})^{+}\ \right] = \mathbf{E}\left[\ e^{-Y}\ \right]\ \chi^2_n\left(y-\epsilon; \nu, \frac{\tilde{\lambda}}{1+2\eta}, \frac{\eta}{1+2\eta} \right) \\ - e^{-y}\ \chi^2_n(y-\epsilon; \nu, \tilde{\lambda}, \eta)\end{aligned} \tag{3.73}$$

the value of an exchange option on e^{-Y} *and* $e^{-Y'}$ *is:*

$$\begin{aligned}\mathbf{E}\left[\ (e^{-Y} - e^{-Y'})^{+}\ \right] = \mathbf{E}\left[\ e^{-Y}\ \right]\ \chi^2_n\left(\epsilon'-\epsilon; \nu, \frac{\tilde{\lambda}}{1+2\eta}, \frac{\eta-\eta'}{1+2\eta} \right) \\ - \mathbf{E}\left[\ e^{-Y'}\ \right]\ \chi^2_n\left(\epsilon'-\epsilon; \nu, \frac{\tilde{\lambda}}{1+2\eta'}, \frac{\eta-\eta'}{1+2\eta'} \right).\end{aligned} \tag{3.74}$$

Proof: The expectation (3.70) and the moment generating function (3.72) are well-known (see e.g. Johnson and Kotz (1970)), the statement (3.71) follows directly from the definition of the probability distribution function. For a proof of (3.73) and (3.74) see Jamshidian (1996).

□

3.8.3 Factor Distributions

As observed by CIR (1985a; 1985b), the square-root dynamics of the factors x_i give rise to noncentral chi-square distributed final values.

Lemma 3.13
Let x *be given by*

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}dW. \tag{3.75}$$

Then $x(T)$ *given* $x(t)$ *is ANC distributed with weight*

$$\eta = \frac{\sigma^2}{4\beta}(1 - e^{-\beta(T-t)})$$

and noncentrality $\tilde{\lambda}$ *and degrees of freedom* ν

$$\tilde{\lambda} = x(t)\frac{4\beta e^{-\beta(T-t)}}{\sigma^2(1 - e^{-\beta(T-t)})} \qquad \nu = \frac{4\alpha}{\sigma^2}.$$

The distribution of the factors $x_i(T)$ remains of the ANC type even under a change of measure to a T-forward measure. This change of measure will be necessary to eliminate discounting with the factors later on, it is defined in lemma 3.14 which is Girsanov's theorem (points (i) and (ii)) combined with a slight extension of results by Jamshidian (1987; 1996) (poins (iii) and (iv)).

Lemma 3.14

Let x follow a CIR-type square-root process under the measure P:

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}\, dW, \tag{3.76}$$

such that $x = 0$ is an unattainable boundary ($\alpha > \frac{1}{2}\sigma^2$) and let $c > 0$ be a positive real number.

(i) Then there is an equivalent probability measure $\tilde{P}_c$, whose restriction on $\mathcal{F}_t$ has the following Radon-Nikodym density w.r.t. P

$$\frac{d\tilde{P}_{c_t}}{dP_t} := Z(t) = \mathcal{E}\left(-\int_0^t \sigma(s)c\sqrt{x(s)}H_{2x}(T-s,c)dW(s)\right).$$

and under which the process $\tilde{W}_t^c$

$$dW_t = d\tilde{W}_t^c - H_{2x}(T-t,c)c\sigma\sqrt{x}dt.$$

is a $\tilde{P}_c$-Brownian motion.

(ii) Expectations under P are transformed to expectations under $\tilde{P}_c$ via

$$\mathbf{E}^P\left[\, e^{-\int_t^T cx(s)ds}F(x(T)) \,\middle|\, \mathcal{F}_t\right] = G(x(t),t,T,c)\mathbf{E}^{\tilde{P}_c}\left[\, F(x(T)) \mid \mathcal{F}_t\,\right] \tag{3.77}$$

where

$$G(x,t,T,c) = \mathbf{E}\left[\, e^{-\int_t^T cx(s)ds} \,\middle|\, \mathcal{F}_t\right] = H_{1x}(T-t,c)e^{-H_{2x}(T-t,c)cx}. \tag{3.78}$$

(iii) Under $\tilde{P}_c$ the process x has the dynamics

$$dx = [\alpha - (\beta + H_{2x}(T-t,c)c\sigma^2)x]dt + \sigma\sqrt{x}\, d\tilde{W}^c, \tag{3.79}$$

and $x(T)$ given $x(t)$ is ANC distributed under $\tilde{P}_c$ with weight η_T

$$\eta_T = \frac{c\sigma^2}{4} H_{2x}(T-t,c) \tag{3.80}$$

and ν_T degrees of freedom and noncentrality parameter $\tilde{\lambda}_T$:

$$\nu_T = \frac{4\alpha}{\sigma^2} \tag{3.81}$$

$$\tilde{\lambda}_T = \frac{4}{\sigma^2} \frac{\frac{\partial}{\partial T} H_{2x}(T-t,c)}{H_{2x}(T-t,c)} x(t). \tag{3.82}$$

(iv) The dynamics of the other factors are unaffected.

Proof: See appendix B.1

□

The lemma also holds for time-dependent parameters with $\alpha(t)/\sigma^2(t) > 1/2$, but here we only need the time-independent case.

The change of measure to P_c^Z removes the discounting with $e^{-\int_0^T cx(t)dt}$, while changing the distribution of $x(T)$ to a ANC distribution with parameters given in equations (3.80) to (3.82).

3.9 Credit Derivatives in the CIR Model

Using the results from section 3.8 we are now able to give solutions for the general pricing formulae for credit derivatives that were derived in section 3.4.

3.9.1 Default Digital Payoffs

In section 3.4.2 the price for a default digital put was given in equation (3.22) as

$$D = \int_0^T \mathbf{E}\left[\lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds} \right] dt.$$

The resulting price for the default digital put is:

Proposition 3.15
The price of a default digital put with maturity T and payoff 1 at default is in the CIR model framework

$$D = \int_0^T \mathbf{E}\left[\,\lambda(t)e^{-\int_0^t \lambda(s)ds}e^{-\int_0^t r(s)ds}\,\right]dt \tag{3.83}$$

$$= \int_0^T \left(\sum_{i=1}^n \overline{w}_i(w_i+\overline{w}_i)(\alpha_i H_{2i}(t,w_i+\overline{w}_i) + \frac{\partial H_{2i}(t,w_i+\overline{w}_i)}{\partial t}x_i(0))\right)\prod_{j=1}^n \overline{B}_{j0}(0,t)dt. \tag{3.84}$$

where for $1 \le j \le n$

$$\overline{B}_{j0}(0,t) = H_{1j}(t,w_j+\overline{w}_j)e^{-H_{2j}(t,w_j+\overline{w}_j)\,(w_j+\overline{w}_j)x_j}. \tag{3.85}$$

Proof: See appendix B.2 on page 126.

□

3.9.2 Credit Derivatives with Option Features

Using the results of lemma 3.4 we can derive the price of a credit spread put option in the CIR framework. Here, according to equation (3.40), we have to evaluate

$$D_t = \mathbf{E}\left[\,e^{-\int_t^{T_1} r(s)+\lambda(s)ds}B(T_1,T_2)(\bar{S}-\widetilde{P}(T_1,T_2))^+\,\Big|\,\mathcal{F}_t\right].$$

The result is given in the following proposition:

Proposition 3.16
(i) The price of a credit spread option with maturity T_1, strike spread $\bar{s}$ and knockout at default on a defaultable zero coupon bond with maturity T_2 is (in the CIR model specification)

$$D^{CSP} = \overline{B}_0(t,T_1)\left\{\mathbf{E}\left[\,e^{-Y}\,\right]\chi_n^2\left(\epsilon'-\epsilon;\nu,\frac{\tilde{\lambda}}{1+2\eta},\frac{\eta-\eta'}{1+2\eta}\right)\right.$$
$$\left.-\mathbf{E}\left[\,e^{-Y'}\,\right]\chi_n^2\left(\epsilon'-\epsilon;\nu,\frac{\tilde{\lambda}}{1+2\eta'},\frac{\eta-\eta'}{1+2\eta'}\right)\right\}. \tag{3.86}$$

(ii) The price of a put option with maturity T_1, strike $\bar{S} = e^{-y}$ and knockout at default on a defaultable zero coupon bond with maturity T_2 is (in the CIR model specification)

$$D^{Put} = \overline{B}_0(t,T_1)\left\{e^{-y}\,\chi_n^2\left(\epsilon' - y; \nu, \tilde{\lambda}, -\eta'\right)\right.$$
$$\left. -\mathbf{E}\left[\,e^{-Y'}\,\right]\,\chi_n^2\left(\epsilon' - y; \nu, \frac{\tilde{\lambda}}{1+2\eta'}, -\frac{\eta'}{1+2\eta'}\right)\right\}. \qquad (3.87)$$

The variables in these formulae are:

$$Y \sim \chi_n^2(\cdot; \nu, \tilde{\lambda}; \eta, \epsilon) \qquad (3.88)$$
$$Y' \sim \chi_n^2(\cdot; \nu, \tilde{\lambda}; \eta', \epsilon') \qquad (3.89)$$
$$\nu_i = \frac{4\alpha_i}{\sigma_i^2} \qquad (3.90)$$
$$\tilde{\lambda}_i = \frac{4\frac{\partial}{\partial T}H_{2i}(T_1 - t, w_i + \overline{w}_i)}{\sigma_i^2 H_{2i}(T_1 - t, w_i + \overline{w}_i)}x_i(t) \qquad (3.91)$$
$$\epsilon = -\ln\bar{S} - \sum_{i=1}^{n}\ln H_{1i}(T_2 - T_1, w_i) \qquad (3.92)$$
$$\eta_i = \frac{\sigma_i^2}{4}w_i(w_i + \overline{w}_i)H_{2i}(T_2 - T_1, w_i)H_{2i}(T_1 - t, w_i + \overline{w}_i) \qquad (3.93)$$
$$\epsilon' = -\sum_{i=1}^{n}\ln H_{1i}(T_2 - T_1, w_i + q\overline{w}_i) \qquad (3.94)$$
$$\eta_i' = \frac{\sigma_i^2}{4}(w_i + \overline{w}_i)(w_i + q\overline{w}_i)H_{2i}(T_2 - T_1, w_i + q\overline{w}_i)H_{2i}(T_1 - t, w_i + \overline{w}_i) \qquad (3.95)$$

and $\mathbf{E}\left[\,e^{-Y}\,\right]$ *is defined in equation* (3.72).

Proof: See appendix B.3 on page 128.

□

In these pricing formulae there are negative signs in front of the last arguments of the ANC distribution function $\chi_n^2(\cdot)$. This is not a problem because $\chi_n^2(\cdot)$ is well-defined as long as these parameters are still larger than $-\frac{1}{2}$. For practical applications this is usually the case, because the η_i and η_i' contain a factor in σ_i^2 (which is very small) and the other factors in η_i and η_i' are not large for a realistic specification.

3.10 Conclusion

In this chapter we demonstrated the Cox process approach to the pricing of credit risk derivatives, and some closed form solutions were given for the case of a multifactor CIR-specification of the credit spread and interest rate dynamics.

The Cox process modelling approach derives its tractability from the fact that the pricing of derivatives can be done in stages:

First, by conditioning on the realisation of the intensity process λ the pricing problem can be reduced to the fairly straightforward case of inhomogeneous Poisson processes, for which most expected values are easily calculated. Thus the explicit reference to default events can be eliminated and replaced by expressions in the intensity of the default process. After this stage the methods of default-free interest rate theory can be applied, because the reference to jump processes has disappeared.

We used specifically the change of measure technique in several dimensions to remove discount factors of the form $\exp - \int_0^T g(s)ds$ where g is an 'interest-rate like' process, and then some well-known properties of the final distributions of the factor processes. In most cases we needed to change the measure to remove the discount factor

$$e^{-\int_0^T r(t)+\lambda(t)dt}$$

which typically arose from expectations of the form

$$\mathbf{E}\left[e^{-\int_0^T r(t)dt} \mathbf{1}_{\{\tau>T\}} X \right] = \mathbf{E}\left[e^{-\int_0^T r(t)+\lambda(t)dt} X \right],$$

i.e. survival contingent payoffs. The resulting measure can be termed a *survival contingent measure.*

The change of measure *after* the elimination of the default indicator function makes sure that the new measure remains equivalent to the original martingale measure. Simply taking a defaultable bond as numeraire would mean that this property is lost, and the change of measure would become much more complicated because the price path of a defaultable bond can be discontinuous.

Even when no closed form solutions are available the conditioning technique is still very useful as pre-processing procedure for a subsequent numerical solution of the pricing equation. Especially Monte Carlo methods are notoriously slow to converge for low-probability events (like defaults), the results can be speeded up significantly if the default events have been removed. P.d.e. methods (like finite differences solvers) are usually based on a Feynman-Kac representation of the price-expectation as an expectation of a stochastic integral and a final payoff in terms of *diffusion processes*. Therefore they cannot handle discrete default events directly[18], a pre-processing to remove the jumps is necessary here, too. The only case when this does not make sense is if one would like to recover the full distribution of the payoffs from the numerical scheme, including default events.

It was seen furthermore that the analogy to the default-free interest rate world does not carry through completely. There is the simple representation (3.10) of a defaultable bond price as 'bond price expectation' with defaultable interest rates given by $\overline{r} = r + q\lambda$ but this does *not* mean that options on defaultable bonds can also be treated as if there were no defaults but just a new interest rate $\overline{r}$.

Most of the results of this chapter are independent of the specification used for the interest-rate and credit spread processes. The transfer of the techniques demonstrated for the Gaussian and CIR specification to another specification should not be too hard.

Another point to mention is the amount of data and information that is needed to price credit risk derivatives. One of the advantages of the Duffie-Singelton setup appeared to be the fact, that default intensities λ and default loss rates q only appeared linked as a product in the bond pricing equation, and the product $q\lambda$ could therefore be estimated directly from the credit spreads — without any further need for separate default and recovery information. This breaks down when credit risk derivatives are considered, the default intensity λ *does* appear separated from the default losses q, both pieces of information are needed separately.

In a practical implementation much of this data will not be available and the financial engineer will have to resort to historical data, market data from other issuers of the same industry, region and rating class, or to fundamental analysis. On the other hand, every derivative needs an underlying, and this underlying should be fairly immune to market manipulation[19]. This means that products with option features are viable if and only if there is some degree of liquidity in the market for the underlying defaultable bond. Liquidity can usually only be expected for large issuers and this means better quality on historical price data and probably liquid markets for further defaultable bonds by the same

[18]It is possible to incorporate jump processes in a finite-difference scheme, but at the cost of computing speed and having to solve matrix inversion problems for full matrices.

[19]See Schönbucher and Wilmott (1993; 1996; 1999) for a detailed discussion of option pricing and hedging in illiquid markets.

issuer that can be used to build a credit spread curve. Therefore CRDs will only be viable in markets and for issuers where the data problems are less severe anyway. Nevertheless, data availability and quality is one of the most pressing problem in the field at the moment.

Apart from data problems, market incompleteness is a further large problem that has to be addressed in future research. With uncertainty about the recovery rates the markets for defaultable bonds are incomplete, and perfect hedges are possible only in exceptional circumstances. The pricing formulae that were derived here depend on the input of a pre-specified martingale measure that should be the output of a thorough analysis of the possibility of hedging and risk-management in credit markets. On the other hand, while being a problem for the pricing and hedging of credit derivatives, market incompleteness has also been one of the major reasons for the success of these instruments: credit derivatives are an important step towards the completion of the credit markets.

Appendix

Appendix A

Calculations to the Gaussian Model

A.1 Proof of Lemma 3.3

Proof: (of Lemma 3.6)
The survival probability is defined as

$$P(0,T) = \mathbf{E}\left[\, \mathbf{1}_{\{\tau > T\}} \,\right]. \tag{A.1}$$

Using iterated expectations and the Cox process properties this can be expanded to

$$\begin{aligned} P(0,T) &= \mathbf{E}\left[\, \mathbf{E}\left[\, \mathbf{1}_{\{\tau > T\}} \,\middle|\, \lambda(t),\ t \geq 0 \,\right] \,\right] \\ &= \mathbf{E}\left[\, \exp\{-\int_0^T \lambda(t)dt\} \,\right]. \end{aligned} \tag{A.2}$$

From assumption 3.5 we know the dynamics of the defaultable and default-free bond prices:

$$\begin{aligned} \frac{dB(t,T)}{B(t,T)} &= r(t)dt + a(t,T)dW \\ \frac{d\overline{B}(t,T)}{\overline{B}(t,T)} &= (r(t) + q\lambda(t))dt + \overline{a}(t,T)dW. \end{aligned}$$

Thus the bond prices satisfy for all $t \leq T_1 \leq T$ (conditional on survival):

$$B(T_1,T) = B(t,T)\exp\left\{ \int_t^{T_1} r(s)ds - \frac{1}{2}\int_t^{T_1} a^2(s,T)ds + \int_t^{T_1} a(s,T)dW_s \right\} \tag{A.3}$$

$$\overline{B}(T_1,T) = \overline{B}(t,T)\exp\left\{\int_t^{T_1} r(s) + q\lambda(s)ds - \frac{1}{2}\int_t^{T_1} \overline{a}(s,T)^2 ds + \int_t^{T_1} \overline{a}(s,T)dW_s\right\}. \tag{A.4}$$

Using the previous equations with $T_1 = T$, $t = 0$ and $B(T,T) = \overline{B}(T,T) = 1$ we reach

$$1 = B(0,T)\exp\left\{\int_0^T r(s)ds - \frac{1}{2}\int_0^T a^2(s,T)ds + \int_0^T a(s,T)dW_s\right\}$$
$$1 = \overline{B}(0,T)\exp\left\{\int_0^T r(s) + q\lambda(s)ds - \frac{1}{2}\int_0^T \overline{a}(s,T)^2 ds + \int_0^T \overline{a}(s,T)dW_s\right\},$$

which can be solved for the integral of λ to yield

$$\begin{aligned}\exp\{-\int_0^T \lambda(s)ds\} = {} & \left(\frac{\overline{B}(0,T)}{B(0,T)}\right)^{\frac{1}{q}} \mathcal{E}\left(\int_0^T \frac{1}{q}(\overline{a}(s,T) - a(s,T))dW_s\right) \\ & \exp\left\{-\frac{1}{2q^2}\int_0^T (\overline{a}(s,T) - a(s,T))[(1+q)a(s,T) - (1-q)\overline{a}(s,T))]ds\right\}.\end{aligned} \tag{A.5}$$

Taking expectations of (A.5) yields equation (3.48).

It remains to show the dynamics (3.46) and (3.47) of the survival probability. From representation (A.2) follows, that

$$M_t := P(t,T)e^{-\int_0^t \lambda(s)ds}$$

is a martingale. Thus $\mathbf{E}[\, dM \,] = 0$ and combined with Itô's lemma this means that

$$\mathbf{E}[\, dP(t,T) \,] = \lambda(t)P(t,T)dt. \tag{A.6}$$

The volatility of $P(t,T)$ can be derived from (3.48) by substituting for the bond prices $B(t,T)$ and $\overline{B}(t,T)$ from equations (A.3) and (A.4) respectively, and using Itô's lemma again. Finally, similar to equations (A.3) and (A.4), the survival probability can be represented as follows for all $t \le T_1 \le T$:

$$P(T_1,T) = P(t,T)\exp\left\{\int_t^{T_1} \lambda(s)ds - \frac{1}{2}\int_t^{T_1} a^p(s,T)^2 ds + \int_t^{T_1} a^p(s,T)dW_s\right\}. \tag{A.7}$$

□

A.2 Proof of Lemma 3.4

The proof of this lemma is an application of Girsanov's theorem.

Proof: (of Lemma 3.7)
First, the expectation has to be converted using iterated expectations and the Cox process properties of the default process:

$$\begin{aligned}\mathbf{E}^Q\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}}X \,\right] &= \mathbf{E}^Q\left[\, \mathbf{E}^Q\left[\, e^{-\int_t^T r(s)ds}\mathbf{1}_{\{\tau>T\}}X \,\middle|\, \lambda(s)\ s\geq 0\right]\,\right] \\ &= \mathbf{E}^Q\left[\, e^{-\int_t^T r(s)+\lambda(s)ds}X\,\right].\end{aligned}$$

Now we use the representation of the default-free bond price (A.3) and the survival probability in (A.7) to substitute for $e^{\int r(s)ds}$ and $e^{\int \lambda(s)ds}$:

$$\begin{aligned}\mathbf{E}^Q\left[\, \beta_{t,T}\mathbf{1}_{\{\tau>T\}}X \,\right] &= B(t,T)P(t,T)\exp\{-\frac{1}{2}\int_t^T a(s,T)^2 + a^p(s,T)^2 ds\} \\ &\quad \mathbf{E}^Q\left[\, \exp\{\int_t^T (a(s,T)+a^p(s,T))dW_s\}X\,\right] \\ &= B(t,T)P(t,T)\exp\{\int_t^T a(s,T)a^p(s,T)ds\} \\ &\quad \mathbf{E}^Q\left[\, \mathcal{E}\left(\int_t^T (a(s,T)+a^p(s,T))dW_s\right)X\,\right]\end{aligned}$$

The rest of the lemma follows directly from Girsanov's theorem. The representation of the zero-recovery defaultable bond price follows by setting $X=1$.

□

A.3 Proof of Proposition 3.4

Proof: (of Proposition 3.8)
We have to calculate

$$x(t) := \mathbf{E}\left[\, \lambda(t)e^{-\int_0^t \lambda(s)ds}e^{-\int_0^t r(s)ds}\,\right].$$

The change of measure to the survival contingent measure P_S (see lemma 3.7, the measure is contingent on survival until t) reduces the problem to finding

$$x(t) = B(0,t)P(0,t)\exp\{\int_0^t a(s,t)a^p(s,t)ds\}\,\mathbf{E}^{P_S}\,[\,\lambda(t)\,] \tag{A.8}$$

where under P_S

$$dW_s^{P_S} = dW_s - (a(s,t) + a^p(s,t))ds$$

is a P_S-Brownian motion.

To evaluate equation (A.8) we need to find the dynamics of λ. Define the forward default intensity

$$\lambda(t,T) := -\frac{\partial}{\partial T}\ln P(t,T).$$

and denote its dynamics with

$$d\lambda(t,T) = \alpha^p(t,T)dt + \sigma^p(t,T)dW_t.$$

The spot default intensity is $\lambda(t) = \lambda(t,t)$. The volatility $\sigma^p(t,T)$ and the drift $\alpha^p(t,T)$ of $\lambda(t,T)$ can be derived directly from equation (A.7):

$$\begin{aligned}\sigma^p(t,T) &= -\frac{\partial}{\partial T}a^p(t,T)\\ \alpha^p(t,T) &= -\sigma^p(t,T)a^p(t,T).\end{aligned}$$

This yields

$$\begin{aligned}\lambda(t) = \lambda(t,t) &= \lambda(0,t) + \int_0^t \alpha^p(s,t)ds + \int_0^t \sigma^p(s,t)dW_s\\ &= \lambda(0,t) + \int_0^t \alpha^p(s,t)ds + \int_0^t \sigma^p(s,t)dW_s^{P_S}\\ &+ \int_0^t (a(s,t) + a^p(s,t))\sigma^p(s,t)ds\\ &= \lambda(0,t) + \int_0^t a(s,t)\sigma^p(s,t)ds + \int_0^t \sigma^p(s,t)dW_s^{P_S}\end{aligned}$$

and therefore

$$\mathbf{E}^{P_S}\,[\,\lambda(t)\,] = \lambda(0,t) + \int_0^t a(s,t)\sigma^p(s,t)ds.$$

$\square$

A.4 Proof of Proposition 3.5

Proof: (of Proposition 3.9)
The proof takes place in three steps: First, we analyse the 'in-the-money' event, then the two parts of the payoff are valued using standard option pricing methods.

Let A be the event that the option is in the money, i.e. $\bar{S}B(T_1,T_2) > \overline{B}(T_1,T_2)$. Using the bond price representations (A.3), (A.4) and (A.7) this event is equivalent to

$$\begin{aligned}&\ln\frac{\bar{S}B(t,T_2)}{\overline{B}(t,T_2)} + \frac{1}{2}\int_t^{T_1} \overline{a}(s,T_2)^2 - a(s,T_2)^2 ds\\ >&q\int_t^{T_1}\lambda(s)ds + \int_t^{T_1}\overline{a}(s,T_2) - a(s,T_2)dW_s\end{aligned}$$

or

$$\begin{aligned}&\ln\frac{\bar{S}B(t,T_2)}{\overline{B}(t,T_2)} + q\ln P(t,T_1) + \frac{1}{2}\int_t^{T_1} \overline{a}(s,T_2)^2 - a(s,T_2)^2 - qa^p(s,T_1)^2 ds\\ >&q\int_t^{T_1} a^p(s,T_2) - a^p(s,T_1)dW_s,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\ln\bar{S} + \ln\frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln\frac{\overline{B}(t,T_2)}{B(t,T_2)}\\ &-\frac{1}{2}\int_t^{T_1}\overline{a}(s,T_1)^2 - a(s,T_1)^2\,ds + \frac{1}{2}\int_t^{T_1}\overline{a}(s,T_2)^2 - a(s,T_2)^2\,ds\\ >&q\int_t^{T_1}\left(a^p(s,T_2) - a^p(s,T_1)\right)dW_s.\end{aligned}$$

Using the 'In the Money' event A the option's payoff can be decomposed:

$$\begin{aligned}&\mathbf{E}\left[\, e^{-\int_t^{T_1} r(s)+\lambda(s)ds}(\bar{S}B(T_1,T_2) - \overline{B}(T_1,T_2))^+ \,\middle|\, \mathcal{F}_t\right]\\ =&\bar{S}\mathbf{E}\left[\, e^{-\int_t^{T_1} r(s)+\lambda(s)ds}B(T_1,T_2)\mathbf{1}_{\{A\}} \,\middle|\, \mathcal{F}_t\right] - \mathbf{E}\left[\, e^{-\int_t^{T_1} r(s)+\lambda(s)ds}\overline{B}(T_1,T_2)\mathbf{1}_{\{A\}} \,\middle|\, \mathcal{F}_t\right]\end{aligned}$$

We now calculate the first term in this expression.

$$I_1 := e^{-\int_t^{T_1} r(s)+\lambda(s)ds}B(T_1,T_2)\mathbf{1}_{\{A\}}$$

$$
\begin{aligned}
=& B(t,T_2)P(t,T_1) \times \exp\Big\{\int_t^{T_1} a(s,T_2)a^p(s,T_1)ds\Big\} \\
& \times \exp\Big\{-\frac{1}{2}\int_t^{T_1}(a(s,T_2)+a^p(s,T_1))^2 ds + \int_t^{T_1} a(s,T_2)+a^p(s,T_1)dW_s\Big\}\mathbf{1}_{\{A\}} \\
=& B(t,T_2)P(t,T_1) \times \exp\Big\{\int_t^{T_1} a(s,T_2)a^p(s,T_1)ds\Big\} \\
& \times \mathcal{E}\Big(\int_t^{T_1} a(s,T_2)+a^p(s,T_1)dW_s\Big)\mathbf{1}_{\{A\}}
\end{aligned}
$$

As the volatility functions are deterministic in the Gaussian setup all we need to evaluate now is $\mathbf{E}\left[\ \mathcal{E}(\int_t^{T_1} a(s,T_2)+a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}}\ \middle|\ \mathcal{F}_t\right]$. From the Girsanov theorem we know that there is a measure $\tilde{P}$ such that

$$
\widetilde{W}_t = W_t - \int_0^t a(s,T_2)+a^p(s,T_1)ds
$$

is a Brownian Motion under $\tilde{P}$, and

$$
d\tilde{P} = \mathcal{E}(\int_t^{T_1} a(s,T_2)+a^p(s,T_1)dW_s)dP.
$$

Thus

$$
\begin{aligned}
& \mathbf{E}^P\left[\ \mathcal{E}(\int_t^{T_1} a(s,T_2)+a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}}\ \middle|\ \mathcal{F}_t\right] \\
=& \mathbf{E}^{\tilde{P}}\left[\ \mathbf{1}_{\{A\}}\ \middle|\ \mathcal{F}_t\right] = \tilde{\mathbf{P}}[\,A\,].
\end{aligned}
$$

The event A is defined as

$$
\begin{aligned}
& \ln\bar{S} + \ln\frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln\frac{\overline{B}(t,T_2)}{B(t,T_2)} \\
& -\frac{1}{2}\int_t^{T_1}\overline{a}(s,T_1)^2 - a(s,T_1)^2\,ds + \frac{1}{2}\int_t^{T_1}\overline{a}(s,T_2)^2 - a(s,T_2)^2\,ds \\
>& q\int_t^{T_1}\left(a^p(s,T_2)-a^p(s,T_1)\right)dW_s \Leftrightarrow \\
& \ln\bar{S} + \ln\frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln\frac{\overline{B}(t,T_2)}{B(t,T_2)} \\
& + \int_t^{T_1} -\frac{1}{2}\overline{a}(s,T_1)^2 + \frac{1}{2}a(s,T_1)^2 + \frac{1}{2}\overline{a}(s,T_2)^2 - \frac{1}{2}a(s,T_2)^2 \\
& - q(a(s,T_2)+a^p(s,T_1))(a^p(s,T_2)-a^p(s,T_1))\,ds
\end{aligned}
$$

$$>q \int_t^{T_1} a^p(s,T_2) - a^p(s,T_1) d\tilde{W}_s$$

Simple calculations now yield that under the measure $\tilde{P}$, the event A has the probability

$$N(d_1)$$

where

$$\begin{aligned} d_1 &= \frac{K + F + \frac{1}{2}V}{\sqrt{V}} \\ K &= \ln \bar{S} + \ln \frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln \frac{\overline{B}(t,T_2)}{B(t,T_2)} \\ F &= \int_t^{T_1} q a^p(s,T_1)\big[a(s,T_2) - a(s,T_1) \\ &\quad - (1-q)(a^p(t,T_2) - a^p(t,T_1))\big]\, ds \\ V &= \int_t^{T_1} q^2 (a^p(s,T_2) - a^p(s,T_1))^2\, ds \end{aligned}$$

The second term:
The second term in the decomposed payoff is valued along the same lines. First we prepare the Girsanov transformation:

$$I_2 := e^{-\int_t^{T_1} r(s)+\lambda(s)ds}\overline{B}(T_1,T_2)\mathbf{1}_{\{A\}} \tag{A.9}$$

$$= \mathbf{1}_{\{A\}}\overline{B}(t,T_2)e^{-(1-q)\int_t^{T_1}\lambda(s)ds}\exp\{-\frac{1}{2}\int_t^{T_1}\overline{a}(s,T_2)^2 ds + \int_t^{T_1}\overline{a}(s,T_2)dW_s\}$$

$$= \mathbf{1}_{\{A\}}\overline{B}(t,T_2)P(t,T_1)^{1-q} \tag{A.10}$$

$$\exp\{-\frac{1}{2}\int_t^{T_1}\overline{a}(s,T_2)^2 + (1-q)a^p(s,T_1)^2 ds \tag{A.11}$$

$$+ \int_t^{T_1}\overline{a}(s,T_2) + (1-q)a^p(s,T_1)dW_s\}$$

$$= \mathbf{1}_{\{A\}}\overline{B}(t,T_2)P(t,T_1)^{1-q} \tag{A.12}$$

$$\exp\{\int_t^{T_1}(1-q)a^p(s,T_1)\big[\overline{a}(s,T_2) - \frac{1}{2}q\overline{a}(s,T_1)\big]ds\} \tag{A.13}$$

$$+ \mathcal{E}\Big(\int_t^{T_1}\overline{a}(s,T_2) + (1-q)a^p(s,T_1)dW_s\Big) \tag{A.14}$$

Girsanov: There is a measure $\tilde{P}$ such that

$$\widetilde{W}_t = W_t - \int_0^t \overline{a}(s,T_2) + (1-q)a^p(s,T_1)ds$$

is a Brownian Motion under $\tilde{P}$, and

$$d\tilde{P} = \mathcal{E}(\int_t^{T_1} \overline{a}(s,T_2) + (1-q)a^p(s,T_1)dW_s)\, dP.$$

Thus

$$\begin{aligned}
&\mathbf{E}^P \left[\mathcal{E}(\int_t^{T_1} \overline{a}(s,T_2) + (1-q)a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}} \;\middle|\; \mathcal{F}_t \right] \\
=&\mathbf{E}^{\tilde{P}} \left[\mathbf{1}_{\{A\}} \;\middle|\; \mathcal{F}_t \right] = \tilde{\mathbf{P}}\,[\, A \,]\,.
\end{aligned}$$

The event A is defined as

$$\begin{aligned}
&\ln \bar{S} + \ln \frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln \frac{\overline{B}(t,T_2)}{B(t,T_2)} \\
&- \frac{1}{2}\int_t^{T_1} \overline{a}(s,T_1)^2 - a(s,T_1)^2\, ds + \frac{1}{2}\int_t^{T_1} \overline{a}(s,T_2)^2 - a(s,T_2)^2\, ds \\
>&q \int_t^{T_1} (a^p(s,T_2) - a^p(s,T_1))\, dW_s \\
\Leftrightarrow& \\
&\ln \bar{S} + \ln \frac{\overline{B}(t,T_1)}{B(t,T_1)} - \ln \frac{\overline{B}(t,T_2)}{B(t,T_2)} \\
&+ \int_t^{T_1} -\frac{1}{2}\overline{a}(s,T_1)^2 + \frac{1}{2}a(s,T_1)^2 + \frac{1}{2}\overline{a}(s,T_2)^2 - \frac{1}{2}a(s,T_2)^2 \\
&- q(\overline{a}(s,T_2) + (1-q)a^p(s,T_1))(a^p(s,T_2) - a^p(s,T_1))\, ds \\
>&q \int_t^{T_1} a^p(s,T_2) - a^p(s,T_1) d\tilde{W}_s.
\end{aligned}$$

After short calculations we reach the probability $\tilde{\mathbf{P}}\,[\, A \,]$ as $N(d_2)$, where d_2 is given by

$$d_2 = d_1 - \sqrt{V}.$$

Combination of the results yields the claim of the proposition.

□

A.5 Proof of Proposition 3.7

The pricing of the plain put option on a defaultable bond is very similar to the pricing of the credit spread put.

Proof: (of Proposition 3.11)
Step 1: The in-the-money event A:

$$A = \{\omega \in \Omega \,|\, \bar{S} > \overline{B}(T_1, T_2)\}$$

A is equivalent to

$$\int_t^{T_1} (\overline{a}(s,T_2) - \overline{a}(s,T_1))dW_s < \ln \frac{\bar{S}\overline{B}(s,T_1)}{\overline{B}(t,T_2)} + \int_t^{T_1} -\frac{1}{2}\overline{a}(s,T_2)^2 + \frac{1}{2}\overline{a}(s,T_1)^2 ds.$$

Step 2: Decomposing the payoff:

$$\begin{aligned} &\mathbf{E}\left[\, e^{-\int_t^{T_1} r(s)+\lambda(s)ds}(\bar{S} - \overline{B}(T_1,T_2))^+ \,\Big|\, \mathcal{F}_t \right] \\ =&\bar{S}\mathbf{E}\left[\, e^{-\int_t^{T_1} r(s)+\lambda(s)ds}\mathbf{1}_{\{A\}} \,\Big|\, \mathcal{F}_t \right] - \mathbf{E}\left[\, e^{-\int_t^{T_1} r(s)+\lambda(s)ds}\overline{B}(T_1,T_2)\mathbf{1}_{\{A\}} \,\Big|\, \mathcal{F}_t \right] \end{aligned}$$

Step 3: The first expectation:

Simplification:

$$I_1 := \bar{S}e^{-\int_t^{T_1} r(s)+\lambda(s)ds}\mathbf{1}_{\{A\}} = \bar{S}\overline{B}_0(t,T_1)\mathcal{E}\Big(\int_t^{T_1} a(s,T_1) + a^p(s,T_1)dW_s\Big)\mathbf{1}_{\{A\}}$$

For this term it remains to evaluate

$$\mathbf{E}\left[\, \mathcal{E}(\int_t^{T_1} a(s,T_1) + a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}} \,\Big|\, \mathcal{F}_t \right].$$

Girsanov:
This is done using a change of measure to the survival contingent measure introduced in

lemma 3.7. There is a measure $\tilde{P}$ such that

$$\widetilde{W}_t = W_t - \int_0^t a(s,T_1) + a^p(s,T_1)ds$$

is a Brownian Motion under $\tilde{P}$, and

$$d\tilde{P} = \mathcal{E}(\int_t^{T_1} a(s,T_1) + a^p(s,T_1)dW_s)dP.$$

Thus

$$\begin{aligned}&\mathbf{E}^P\left[\,\mathcal{E}(\int_t^{T_1} a(s,T_1) + a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}}\,\middle|\,\mathcal{F}_t\right]\\ =&\mathbf{E}^{\tilde{P}}\left[\,\mathbf{1}_{\{A\}}\,\middle|\,\mathcal{F}_t\right] = \tilde{\mathbf{P}}\left[\,A\,\right].\end{aligned}$$

Probability of A under $\tilde{P}$:

$$\begin{aligned}&\int_t^{T_1} (\overline{a}(s,T_2) - \overline{a}(s,T_1)))dW_s\\ =&\int_t^{T_1} (\overline{a}(s,T_2) - \overline{a}(s,T_1)))d\widetilde{W}_s + \int_t^{T_1} (\overline{a}(s,T_2) - \overline{a}(s,T_1))(a(s,T_1) + a^p(s,T_1))ds\\ <&\ln\frac{\bar{S}\overline{B}(t,T_1)}{\overline{B}(t,T_2)} + \int_t^{T_1} -\frac{1}{2}\overline{a}(s,T_2)^2 + \frac{1}{2}\overline{a}(s,T_1)^2ds.\end{aligned}$$

This event has the probability $N(d_1)$ where d_1 is given by

$$d_1 = \frac{K + F_1 + \frac{1}{2}V}{\sqrt{V}}$$

where

$$\begin{aligned}K &= \ln\frac{\bar{S}\overline{B}(t,T_1)}{\overline{B}(t,T_2)}\\ F_1 &= -\int_t^{T_1} (\overline{a}(s,T_2) + a(s,T_1) + a^p(s,T_1))(\overline{a}(s,T_2) - \overline{a}(s,T_1))ds\\ V &= \int_t^{T_1} (\overline{a}(s,T_2) - \overline{a}(s,T_1))^2ds.\end{aligned}$$

Step 4: The second expectation:

Simplification:

I_2 here is of the same form as I_2 in the proof of the credit spread put option. Therefore its value is given by expression (A.14):

$$
\begin{aligned}
I_2 :=& e^{-\int_t^{T_1} r(s)+\lambda(s)ds}\overline{B}(T_1,T_2)\mathbf{1}_{\{A\}} \\
=& \overline{B}(t,T_2)P(t,T_1)^{1-q}\exp\big\{(1-q)\int_t^{T_1} a^p(s,T_1)(\overline{a}(s,T_2)-\frac{1}{2}qa^p(s,T_1))ds\big\} \\
& \mathcal{E}(\int_t^{T_1} \overline{a}(s,T_2)+(1-q)a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}}
\end{aligned}
$$

Girsanov:
The appropriate change of measure also carries through from the credit spread put option. There is a measure $\tilde{P}$ such that

$$
\widetilde{W}_t = W_t - \int_0^t \overline{a}(s,T_2)+(1-q)a^p(s,T_1)ds
$$

is a Brownian Motion under $\tilde{P}$, and

$$
d\tilde{P} = \mathcal{E}(\int_t^{T_1} \overline{a}(s,T_2)+(1-q)a^p(s,T_1)dW_s)dP.
$$

Thus

$$
\begin{aligned}
& \mathbf{E}^P\left[\mathcal{E}(\int_t^{T_1} \overline{a}(s,T_2)+(1-q)a^p(s,T_1)dW_s)\mathbf{1}_{\{A\}} \,\middle|\, \mathcal{F}_t\right] \\
=& \mathbf{E}^{\tilde{P}}\left[\mathbf{1}_{\{A\}} \,\middle|\, \mathcal{F}_t\right] = \tilde{\mathbf{P}}[\,A\,].
\end{aligned}
$$

After short calculations the event A is found to have under $\tilde{P}$ the probability $N(d_2)$, where d_2 is given by

$$
d_2 = \frac{K + F_2 - \frac{1}{2}V}{\sqrt{V}}
$$

where

$$
\begin{aligned}
K &= \ln\frac{\bar{S}\overline{B}(t,T_1)}{\overline{B}(t,T_2)} \\
F_2 &= F_1 + q\int_t^{T_1} (\overline{a}(s,T_2)-\overline{a}(s,T_1))a^p(s,T_1)ds
\end{aligned}
$$

$$V = \int_t^{T_1} (\overline{a}(s, T_2) - \overline{a}(s, T_1))^2 ds.$$

Combining these results yields the claim of the proposition.

□

Appendix B

Calculations to the CIR Model

B.1 Proof of Lemma 3.7

Proof: (of Lemma 3.14)
Consider the process x with the P-dynamics

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}\, dW$$

and the process $y = cx$. Note that y has the dynamics

$$\begin{aligned} dy = d(cx) &= (c\alpha - \beta y)dt + \sigma\sqrt{c}\,\sqrt{y}\, dW \\ &=: (\hat{\alpha} - \hat{\beta} y)dt + \hat{\sigma}\sqrt{y}\, dW, \end{aligned} \tag{B.1}$$

which is still of the CIR square-root form.

(i) and (ii)
The proof of points (i) and (ii) of lemma 3.14 is an application of the Girsanov theorem and the change of measure technique: Define

$$g(0,t) = e^{-\int_0^t y(s)ds}. \tag{B.2}$$

Then

$$Z(t) := \frac{1}{G(y(0),0,T)} g(0,t) G(y(t),t,T) \tag{B.3}$$

is a positive martingale with initial value $Z(0) = 1$ and can therefore be used as a Radon-

Nikodym density for a change of measure from P to $\tilde{P}_c$ defined by $d\tilde{P}_c/dP = Z$, and

$$Z(t) = \mathcal{E}\left(-\int_0^t \sigma c\sqrt{x(s)}H_{2x}(T-s,c)dW(s)\right).$$

Then

$$Z(t)\mathbf{E}^{\tilde{P}_c}\left[\,F(x(T)) \mid \mathcal{F}_t\,\right] = \mathbf{E}^P\left[\,Z(T)F(x(T)) \mid \mathcal{F}_t\,\right] = \mathbf{E}^P\left[\,e^{-\int_t^T cx(s)ds}F(x(T))\,\middle|\,\mathcal{F}_t\right]$$

(iii)
The P dynamics of x are

$$dx = (\alpha - \beta x)dt + \sigma\sqrt{x}\,dW$$

where W is a P-Brownian motion. By Girsanov's theorem

$$dW = d\tilde{W}^c - H_{2x}(T-t,c)c\sigma\sqrt{x}dt$$

where $\tilde{W}^c$ is a $\tilde{P}_c$-Brownian motion. Thus

$$dx = (\alpha - \beta x)dt - H_{2x}(T-t,c)c\sigma^2 x dt + \sigma\sqrt{x}\,d\tilde{W}^c. \tag{B.4}$$

The distribution of $x(T)$ under $\tilde{P}_c$ can be found e.g. in Jamshidian (1996) or Schlögl (1997). There the distribution is only given for $c = 1$, but by applying these results to y equations (3.79)- (3.82) follow directly.

(iv)
The dynamics of the other factors remain unchanged because the change of measure here does not affect them. (Z does not depend on any of the other factors.)

□

B.2 Proof of Proposition 3.8

Proof: (of Proposition 3.15)
We have to evaluate

$$\mathbf{E}\left[\,\lambda(t)e^{-\int_0^t \lambda(s)ds}e^{-\int_0^t r(s)ds}\,\right].$$

First, simplify the expression in the expectation operator:

$$\begin{aligned}
&\lambda(t)e^{-\int_0^t \lambda(s)+r(s)ds} \\
&= \left(\sum_{i=1}^n \overline{w}_i x_i(t)\right) e^{\sum_{j=1}^n - \int_0^t (w_j+\overline{w}_j)x_j(s)ds} \\
&= \sum_{i=1}^n \left(\overline{w}_i x_i(t) e^{\sum_{j=1}^n - \int_0^t (w_j+\overline{w}_j)x_j(s)ds}\right).
\end{aligned}$$

Looking at the i-th summand:

$$\begin{aligned}
&\mathbf{E}\left[\ \overline{w}_i x_i(t) e^{\sum_{j=1}^n - \int_0^t (w_j+\overline{w}_j)x_j(s)ds}\ \right] \\
=&\mathbf{E}\left[\ \overline{w}_i x_i(t) e^{- \int_0^t (w_i+\overline{w}_i)x_i(s)ds}\ \right] \mathbf{E}\left[\ e^{\sum_{j\neq i}^n - \int_0^t (w_j+\overline{w}_j)x_j(s)ds}\ \right] \\
=&\mathbf{E}\left[\ \overline{w}_i x_i(t) e^{- \int_0^t (w_i+\overline{w}_i)x_i(s)ds}\ \right] \prod_{j\neq i} \mathbf{E}\left[\ e^{- \int_0^t (w_j+\overline{w}_j)x_j(s)ds}\ \right] \\
=&\overline{w}_i \mathbf{E}\left[\ x_i(t) e^{- \int_0^t (w_i+\overline{w}_i)x_i(s)ds}\ \right] \prod_{j\neq i} \overline{B}_{j0}(0,t),
\end{aligned}$$

where $\overline{B}_{j0}(0,t)$ is defined as in the proposition.
It remains to evaluate $\mathbf{E}\left[\ \overline{w}_i x_i(t) e^{- \int_0^t (w_i+\overline{w}_i)x_i(s)ds}\ \right]$. For this we change the measure according to lemma 3.14 choosing $c = w_i + \overline{w}_i$. Then

$$\mathbf{E}\left[\ x_i(t) e^{- \int_0^t c x_i(s)ds}\ \right] = \overline{B}_{i0}(0,t)\, \mathbf{E}^{\tilde{P}_c}\left[\ x_i(t)\ \right].$$

By lemma 3.14 (iii) and lemma 3.12 equation (3.70) the expectation of $x_i(t)$ under $\tilde{P}_c$ is

$$\begin{aligned}
&\mathbf{E}^{\tilde{P}_c}\left[\ x_i(t)\ \right] = \eta_t(\nu_t + \tilde{\lambda}_t) \\
=&c\frac{\sigma_i^2}{4} H_{2i}(t,c)\left(\frac{4\alpha_i}{\sigma_i^2} + \frac{4}{\sigma_i^2} \frac{\frac{\partial}{\partial T} H_{2i}(t,c)}{H_{2i}(t,c)} x_i(0)\right) \\
=&c\alpha_i H_{2i}(t,c) + c\frac{\partial}{\partial T} H_{2i}(t,c) x_i(0).
\end{aligned}$$

Thus, combining all yields

$$\begin{aligned}
&\mathbf{E}\left[\ \lambda(t) e^{-\int_0^t \lambda(s)ds} e^{-\int_0^t r(s)ds}\ \right] \\
=&\left(\sum_{i=1}^n \overline{w}_i(\overline{w}_i + w_i)(\alpha_i H_{2i}(t, w_i+\overline{w}_i) + \frac{\partial}{\partial T} H_{2i}(t, w_i+\overline{w}_i) x_i(0))\right) \prod_{j=1}^n \overline{B}_{j0}(0,t).
\end{aligned}$$

□

B.3 Proof of Proposition 3.9

Proof: (of Proposition 3.16)
The expectation, that has to be calculated for the credit spread put, is

$$\begin{aligned} D^{CSP} =& \mathbf{E}\left[\, e^{-\int_t^{T_1}\sum_{i=1}^n (w_i+\overline{w}_i)x_i(s)ds}(\bar{S}B(T_1,T_2)-\overline{B}(T_1,T_2))^+ \,\right] \\ =& \overline{B}_0(t,T_1)\,\mathbf{E}^{\tilde{P}}\left[\,(\bar{S}B(T_1,T_2)-\overline{B}(T_1,T_2))^+\,\right] \end{aligned}$$

where the bond prices are

$$\begin{aligned} B(T_1,T_2) =& \prod_{i=1}^n H_{1i}(T_2-T_1,w_i)e^{-H_{2i}(T_2-T_1,w_i)w_ix_i(T_1)} \\ \overline{B}(T_1,T_2) =& \prod_{i=1}^n H_{1i}(T_2-T_1,w_i+q\overline{w}_i)e^{-H_{2i}(T_2-T_1,w_i+q\overline{w}_i)(w_i+q\overline{w}_i)x_i(T_1)}. \end{aligned}$$

The measure $\tilde{P}$ is the T_1-survival measure which is reached by changing the measure for each component x_i according to lemma 3.14 with $c_i = w_i + \overline{w}_i$. Under $\tilde{P}$ the component $x_i(T_1)$ is $\chi_1^2(\nu_i, \tilde{\lambda}_i; \bar{\eta}_i)$ ANC distributed with ν_i and $\tilde{\lambda}_i$ given in the proposition and

$$\bar{\eta}_i = (w_i + \overline{w}_i)\frac{\sigma_i^2}{4}H_{2i}(T_1 - t, w_i + \overline{w}_i).$$

Then the variable Y in the proposition is defined s.t. e^{-Y} has the same distribution as $\bar{S}B(T_1, T_2)$ under the T_1-survival measure.
The variable Y' is defined to have the distribution of $\overline{B}(T_1, T_2)$ under the T_1-survival measure.

The claim of the proposition now follows directly from equations (3.73) and (3.74) in lemma 3.12. The formula for the Put-option is reached by setting $\eta_i = 0$ and $\epsilon = y$ in the formula for the credit spread put option.

□

Bibliography

Philippe Artzner and Freddy Delbaen. Credit risk and prepayment option. *Astin Bulletin*, 22:81–96, 1992.

Philippe Artzner and Freddy Delbaen. Default risk insurance and incomplete markets. Working paper, Université Luis Pasteur et C.N.R.S., Strasbourg, June 1994.

T. Björk, Y. Kabanov, and W. Runggaldier. Bond market structure in the presence of marked point process. Working paper, Department of Finance, Stockholm School of Economics, S-113 83 Stockholm, Sweden, July 1996.

Tomas Björk, Giovanni Di Masi, Yuri Kabanov, and Wolfgang Runggaldier. Towards a general theory of bond markets. *Finance and Stochastics*, 1(2):141–174, 1997.

Fischer Black and John Cox. Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance*, pages 351–367, 1976.

Fischer Black, Emanuel Derman, and William Toy. A one factore model of interest rates and its application to treasury bond options. *Financial Analysts Journal*, pages 33–39, 1990.

Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–54, 1973.

Alan Brace, Dariusz Gatarek, and Marek Musiela. The market model of interest rate dynamics. *Mathematical Finance*, 7(2):127–155, 1997.

Pierre Brémaud. *Point Processes and Queues*. Springer, Berlin, Heidelberg, New York, 1981.

Eric Briys and Francois de Varenne. Valuing risky fixed rate debt: An extension. *Journal of Financial and Quantitative Analysis*, 32(2):239–248, June 1997.

Ren-Raw Chen and Louis Scott. Interest rate options in multifactor cox-ingersoll-ross models of the term structure. *The Journal of Derivatives*, 3(2):52–72, 6 1995.

John Cox, Jonathan E. Ingersoll, and Stephen A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53:363–384, 1985a.

John Cox, Jonathan E. Ingersoll, and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53:385–407, 1985b.

Sanjiv R. Das and Rangarajan K. Sundaram. A direct approach to arbitrage-free pricing of credit derivatives. Working Paper 6635, National Bureau of Economic Research, July 1998.

Sanjiv Ranjan Das. Credit risk derivatives. *Journal of Derivatives*, 2(3):7–23, Spring 1995.

Sanjiv Ranjan Das and Peter Tufano. Pricing credit-sensitive debt when interest rates, credit ratings and credit spreads are stochastic. Working paper, Harvard Business School, December 1994.

Satyajit Das, editor. *Credit Derivatives: Trading and Management of Credit and Default Risk*. Wiley Frontiers in Finance. John Wiley & Sons, New York, Chichester, Singapore, 1998.

Gregory R. Duffee. Estimating the price of default risk. Working paper, Federal Reserve Board, Washington DC, September 1995.

Darrel Duffie and David Lando. Term structures of credit spreads with incomplete accounting information. Working paper, Graduate School of Business, Stanford University, June 1997.

Darrell Duffie. Forward rate curves with default risk. Working paper, Graduate School of Business, Stanford University, December 1994.

Darrell Duffie. First-to-default valuation. Working paper, Graduate School of Business, Stanford University, 1998.

Darrell Duffie. Credit swap valuation. Working paper, Graduate School of Business, Stanford University, 1999.

Darrell Duffie and Ming Huang. Swap rates and credit quality. Working paper, Graduate School of Business, Stanford University, October 1994.

Darrell Duffie and Ming Huang. Swap rates and credit quality. *Journal of Finance*, 51(3): 921–949, 7 1996.

Darrell Duffie and Rui Kan. A yield-factor model of interest rates. *Mathematical Finance*, 6:379–406, 1996.

Darrell Duffie, Mark Schroder, and Costis Skiadas. Recursive valuation of defaultable securities and the timing of resolution of uncertainty. Working Paper 195, Kellogg Graduate School of Management, Department of Finance, Northwestern University, November 1994. revised March 1995.

Darrell Duffie and Kenneth J. Singleton. An econometric model of the term structure of interest-rate swap yields. *The Journal of Finance*, 52(4):1287–1322, 1997.

Darrell Duffie and Kenneth J. Singleton. Modeling term structures of defaultable bonds. *The Review of Financial Studies*, 12(4):687–720, 1999.

K. Düllmann, M. Uhrig-Homburg, and M. Windfuhr. Risk structure of interest rates: an empirical analysis for Deutschemark-denominated bonds. *to appear in: European Financial Management*, 1999.

David Epstein, Nick Mayor, Philipp J. Schönbucher, and P. Wilmott. The valuation of a firm advertising optimally. *The Quarterly Review of Economics and Finance*, 38(2): 149–166, 1998.

Bjorn Flesaker, Lane Houghston, Laurence Schreiber, and Lloyd Sprung. Taking all the credit. *Risk Magazine*, 7:105–108, 1994.

Julian R. Franks and Walter N. Torous. A comparison of financial recontracting in distressed exchanges and Chapter 11 reorganizations. *Journal of Financial Economics*, 35: 349–370, 1994.

R. Geske. The valuation of corporate liabilities as compound options. *Journal of Financial and Quantitative Analysis*, 12:541–552, 1977.

Richard J. Haber, Philipp J. Schönbucher, and Paul Wilmott. Pricing Parisian options. *Journal of Derivatives*, 6(3):71–79, 1999.

David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60: 77–105, 1992.

Thomas S. Y. Ho and Sang-Bin Lee. Term structure movements and pricing interest rate contingent claims. *Journal of Finance*, 41:1011–1029, 1986.

John Hull and Alan White. One factor interest rate models and the valuation of interest rate derivative securities. *Journal of Financial and Quantitative Analysis*, 28:235–254, 1993.

Jean Jacod and Albert N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, Berlin, Heidelberg, New York, 1988.

Farshid Jamshidian. Pricing of contingent claims in the one-factor term structure model. Working paper, Sakura Global Capital, July 1987.

Farshid Jamshidian. A simple class of square-root interest-rate models. *Applied Mathematical Finance*, 2:61–72, 1995.

Farshid Jamshidian. Bond futures and option evaluation in the quadratic interest rate model. *Applied Mathematical Finance*, 3:93–115, 1996.

Farshid Jamshidian. LIBOR and swap market models and measures. *Finance and Stochastics*, 1(4):293–330, 1997.

Robert Jarrow and Dilip Madan. Option pricing using the term structure of interest rates to hedge systematic discontinuities in asset returns. *Mathematical Finance*, 5(4):311–336, 10 1995.

Robert A. Jarrow, David Lando, and Stuart M. Turnbull. A markov model for the term structure of credit risk spreads. *The Review of Financial Studies*, 10(2):481–523, 1997.

Robert A. Jarrow and Stuart M. Turnbull. Pricing derivatives on financial securities subject to credit risk. *Journal of Finance*, 50:53–85, 1995.

Norman L. Johnson and Samuel Kotz. *Continuous Univariate Distributions*, volume 2. Houghton Mifflin Co., New York, 1970.

David Lando. *Three Essays on Contingent Claims Pricing.* PhD thesis, Graduate School of Management, Cornell University, 1994.

David Lando. On Cox processes and credit risky securities. *Review of Derivatives Research*, 2(2/3):99–120, 1998.

Hayne E. Leland. Risky debt, bond covenants and optimal capital structure. *Journal of Finance*, 49:1213–1252, 1994.

Hayne E. Leland and Klaus Bjerre Toft. Optimal capital structure, endogenous bankrupcy and the term structure of credit spreads. *Journal of Finance*, 51(3):987–1019, 1996.

David X. Li. The valuation of basket credit derivatives. *Credit Metrics Monitor*, 2:34–50, April 1999.

F. Longstaff and E. Schwartz. Interest rate volatility and the term structure: A two factor general equilibrium model. *The Journal of Finance*, 47:1259–82, 1992.

Francis Longstaff and Eduardo Schwartz. The pricing of credit risk derivatives. *Journal of Fixed Income*, 5(1):6–14, June 1995.

Francis A. Longstaff and Eduardo S. Schwartz. A simple approach to valuing risky fixed and floating rate debt. Working Paper 22–93, Anderson Graduate School of Management, University of California, Los Angeles, October 1993. revised November 1994.

Christopher Lotz and Lutz Schlögl. Default risk in a market model. *to appear in: Journal of Banking and Finance*, 1999.

Dilip Madan and Haluk Unal. Pricing the risks of default. *Review of Derivatives Research*, 2(2/3):121–160, 1998.

Pierre Mathieu and Patrick d'Herouville. *Les Dérivés de Crédit.* Politique générale, Finance et Marketing. Economica, Paris, collection gestion edition, 1998.

Pierre Mella-Barral and William R. M. Perraudin. Strategic debt service. *Journal of Finance*, 52:531–556, 1997.

Robert C. Merton. On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance*, 29:449–470, 1974.

Robert C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.

Robert C. Merton. On the pricing of contingent claims and the Modigliani-Miller theorem. *Journal of Financial Economics*, 5:241–249, 1977.

Kristian R. Miltersen, Klaus Sandmann, and Dieter Sondermann. Closed form solutions for term structure derivatives with log-normal interest rates. *The Journal of Finance*, 52(1):409–430, 3 1997.

Izrael Nelken. *Implementing Credit Derivatives*. Irwin Library of Investment and Finance. McGraw-Hill, New York, London, Sydney, 1999.

Yiannos A. Pierides. The pricing of credit risk derivatives. *Journal of Economic Dynamics and Control*, 21:1579–1611, 1997.

P. Protter. *Stochastic Integration and Differential Equations*. Springer, Berlin, Heidelberg, New York, 1990.

Klaus Sandmann and Dieter Sondermann. A note on the stability of lognormal interest rate models and the pricing of eurodollar futures. *Mathematical Finance*, 7(2):119–126, 1997.

Erik Schlögl. *Interest Rate Factor Models: Term Structure Dynamics and Derivatives Pricing*. PhD thesis, University of Bonn, Faculty of Economics, 1997.

Philipp J. Schönbucher. Option pricing and hedging in finitely liquid markets. Master's thesis, University of Oxford, Mathematical Institute, September 1993.

Philipp J. Schönbucher. The term structure of defaultable bond prices. Discussion Paper B-384, University of Bonn, SFB 303, August 1996a.

Philipp J. Schönbucher. Valuation of securities subject to credit risk. Working paper, University of Bonn, Department of Statistics, February 1996b.

Philipp J. Schönbucher. Modelling defaultable bond prices. Working paper, London School of Economics, Financial Markets Group, July 1997a.

Philipp J. Schönbucher. Pricing credit risk derivatives. Working Paper wp-10, London School of Economics, Financial Markets Group, July 1997b.

Philipp J. Schönbucher. Term structure modelling of defaultable bonds. *The Review of Derivatives Studies, Special Issue: Credit Risk*, 2(2/3):161–192, 1998.

Philipp J. Schönbucher. A market model of stochastic implied volatility. *Philosophical Transactions of the Royal Society of London, Series A*, 357(1758):2071–2092, 1999.

Philipp J. Schönbucher and Paul Wilmott. Hedging in illiquid markets: Nonlinear effects. In Oskar Mahrenholtz, Kurt Marti, and Reinhard Mennicken, editors, *ICIAM/GAMM 95: Proceedings of the Third International Congress on Industrial and Applied Mathematics, Hamburg, July 3-7 1995: Issue 3: Applied Stochastics and Optimization*, pages 81–84. Special Issue: Zeitschrift für Angewandte Mathematik und Mechanik (ZAMM), 1996.

Philipp J. Schönbucher and Paul Wilmott. The feedback effect of hedging in illiquid markets. *to appear in: SIAM Journal of Applied Mathematics*, 1999.

Janet M. Tavakoli. *Credit Derivatives: A Guide to Instruments and Applications.* Wiley Series in Financial Engineering. John Wiley & Sons, New York, Chichester, Singapore, 1998.

Oldrich Vasicek. An equilibrium characterisation of the term structure. *Journal of Financial Economics*, 5:177–188, 1977.

Chunsheng Zhou. A jump-diffusion approach to modeling credit risk and valuing defaultable securities. Finance and Economics Discussion Paper Series 1997/15, Board of Governors of the Federal Reserve System, March 1997.