CREDIT RISK MODELING AND VALUATION: AN INTRODUCTION

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Abstract

Credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement. We review the structural, reduced form and incomplete information approaches to estimating joint default probabilities and prices of credit sensitive securities.

Key words: credit risk; default risk; structural approach; reduced form approach; incomplete information approach; intensity; trend; compensator.

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1 Introduction

Credit risk is the distribution of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement. Examples range from agency downgrades to failure to service debt to liquidation. Credit risk pervades virtually all financial transactions.

The distribution of credit losses is complex. At its center is the probability of default, by which we mean any type of failure to honor a financial agreement. To estimate probability of default, we need to specify

- a model of investor uncertainty;
- a model of the available information and its evolution over time; and
- a model definition of the default event.

However, default probabilities alone are not sufficient to price credit sensitive securities. We need, in addition,

- a model for the riskfree interest rate;
- a model of recovery upon default; and
- a model of the premium investors require as compensation for bearing systematic credit risk.

The credit premium maps actual default probabilities to market-implied probabilities that are embedded in market prices. To price securities that are sensitive to the credit risk of multiple issuers and to measure aggregated portfolio credit risk, we also need to specify

- a model that links defaults of several entities.

There are three main quantitative approaches to analyzing credit. In the *structural* approach, we make explicit assumptions about the dynamics of a firm's assets, its capital structure, and its debt and share holders. A firm defaults if its assets are insufficient according to some measure. In this situation a corporate liability can be characterized as an option on the firm's assets. The *reduced form* approach is silent about why a firm defaults. Instead, the dynamics of default are exogenously given through a default rate, or intensity. In this approach, prices of credit sensitive securities can be calculated as if they were default free using an interest rate that is the riskfree rate adjusted by the intensity. The *incomplete information* approach combines the structural and reduced form models. While avoiding their difficulties, it picks the best features of both approaches: the economic and intuitive appeal of the structural approach and the tractability and empirical fit of the reduced form approach.

This article reviews these approaches in the context of the multiple facets of credit modeling that are mentioned above. Our goal is to provide a concise overview and a guide to the large and growing literature on credit risk.

2 Structural credit models

The basis of the structural approach, which goes back to Black & Scholes (1973) and Merton (1974), is that corporate liabilities are contingent claims on the assets of a firm. The market value of the firm is the fundamental source of uncertainty driving credit risk.

2.1 Classical approach

Consider a firm with market value V, which represents the expected discounted future cash flows of the firm. The firm is financed by equity and a zero coupon bond with face value K and maturity date T. The firm's contractual obligation is to repay the amount K to the bond investors at time T. Debt covenants grant bond investors absolute priority: if the firm cannot fulfil its payment obligation, then bond holders will immediately take over the firm. Hence the default time τ is a discrete random variable given by

$$\tau = \begin{cases} T & \text{if} \quad V_T < K \\ \infty & \text{if} \quad \text{else.} \end{cases} \tag{1}$$

Figure 1 depicts the situation graphically.

To calculate the probability of default, we make assumptions about the distribution of assets at debt maturity under the physical probability P. The standard model for the evolution of asset prices over time is geometric Brownian motion:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \quad V_0 > 0, \tag{2}$$

where $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. Setting $m = \mu - \frac{1}{2}\sigma^2$, Ito's lemma implies that

$$V_t = V_0 e^{mt + \sigma W_t}.$$

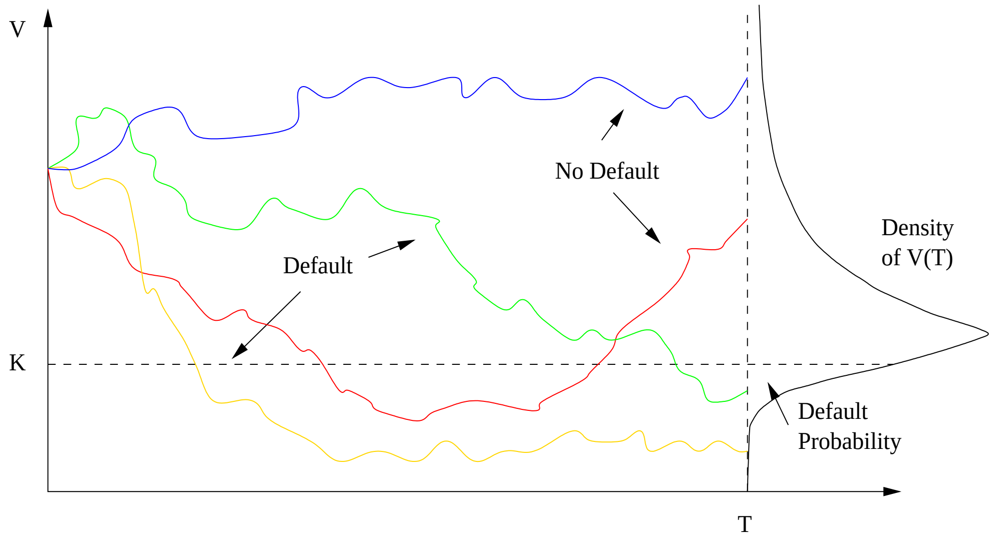

Figure 1: Default in the classical approach.

Since W_T is normally distributed with mean zero and variance T, default probabilities $p(T)$ are given by

$$p(T) = P[V_T < K] = P[\sigma W_T < \log L - mT] = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right)$$

where $L = \frac{K}{V_0}$ is the initial leverage ratio and Φ is the standard normal distribution function.

Assuming that the firm can neither repurchase shares nor issue new senior debt, the payoffs to the firm's liabilities at debt maturity T are as summarized in Table 1. If the asset value V_T exceeds or equals the face value K of the bonds, the bond holders will receive their promised payment K and the shareholders will get the remaining $V_T - K$. However, if the value of assets V_T is less than K, the ownership of the firm will be transferred to the bondholders, who lose the amount $K - V_T$. Equity is worthless because of limited liability. Summarizing, the value of the bond issue B_T^T at time T is given by

$$B_T^T = \min(K, V_T) = K - \max(0, K - V_T).$$

This payoff is equivalent to that of a portfolio composed of a default-free loan with face value K maturing at T and a short European put position on the assets of the firm with strike K and maturity T. The value of the equity E_T

	Assets	Bonds	Equity
No Default	$V_T \geq K$	K	$V_T - K$
Default	$V_T < K$	V_T	0

Table 1: Payoffs at maturity in the classical approach.

at time T is given by

$$E_T = \max(0, V_T - K),$$

which is equivalent to the payoff of a European call option on the assets of the firm with strike K and maturity T.

Pricing equity and credit risky debt reduces to pricing European options. We consider the classical Black-Scholes setting, where riskfree interest rates $r > 0$ are constant and firm assets V follow geometric Brownian motion (2). The equity value is given by the Black-Scholes call option formula:

$$E_0 = V_0 \Phi(d_1) - e^{-rT} K \Phi(d_2) \tag{3}$$

where

$$d_1 = \frac{(r + \frac{1}{2}\sigma^2)T - \log L}{\sigma\sqrt{T}} \qquad \text{and} \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

While riskfree zero coupon bond prices are just $K \exp(-rT)$ with T being the bond maturity, the value of the corresponding credit-risky bonds is

$$B_0^T = K e^{-rT} - P(\sigma, T, K, r, V_0)$$

where P is the Black-Scholes put option formula. We note that the value of the put is just equal to the present value of the default loss suffered by bond investors. This is the discount for default risk relative to the riskfree bond, which is valued at $K \exp(-rT)$. This yields

$$B_0^T = V_0 - V_0 \Phi(d_1) + e^{-rT} K \Phi(d_2)$$

which together with (3) proves the market value identity

$$V_0 = E_0 + B_0^T.$$

While clearly both equity and debt values depend on the firm's leverage ratio, this equation shows that their sum does not. This shows that the Modigliani & Miller (1958) theorem holds also in the presence of default. This result asserts that the market value of the firm is independent of its leverage, see Rubinstein (2003) for a discussion. This is not the case for all credit models as we show below.

The *credit spread* is the difference between the yield on a defaultable bond and the yield an otherwise equivalent default-free zero bond. It gives the excess return demanded by bond investors to bear the potential default losses. Since the yield $y(t,T)$ on a bond with price $b(t,T)$ satisfies $b(t,T) = \exp(-y(t,T)(T-t))$, we have for the credit spread $S(t,T)$ at time t,

$$S(t,T) = -\frac{1}{T-t}\log\left(\frac{B_t^T}{\bar{B}_t^T}\right), \quad T > t, \tag{4}$$

where $\bar{B}_t^T$ is the price of a default-free bond maturing at T. The term structure of credit spreads is the schedule of $S(t,T)$ against T, holding t fixed. In the Black-Scholes setting, we have $\bar{B}_t^T = K\exp(-r(T-t))$ and we obtain

$$S(0,T) = -\frac{1}{T}\log\left(\Phi(d_2) + \frac{1}{L}e^{rT}\Phi(-d_1)\right), \quad T > 0,$$

which is a function of maturity T, asset volatility σ (the firm's business risk), the initial leverage ratio L, and riskfree rates r. Letting leverage be 80% and riskfree rates be 6%, in Figure 2 we plot the term structure of credit spreads for varying asset volatilities.

2.2 First-passage approach

In the classical approach, firm value can dwindle to almost nothing without triggering default. This is unfavorable to bondholders, as noted by Black & Cox (1976). Bond indenture provisions often include safety covenants that give bond investors the right to reorganize a firm if its value falls below a given barrier.

Suppose the default barrier D is a constant valued in $(0, V_0)$. Then the default time τ is a continuous random variable valued in $(0, \infty]$ given by

$$\tau = \inf\{t > 0 : V_t < D\} \tag{5}$$

Figure 3 depicts the situation graphically. In the Black-Scholes setting with

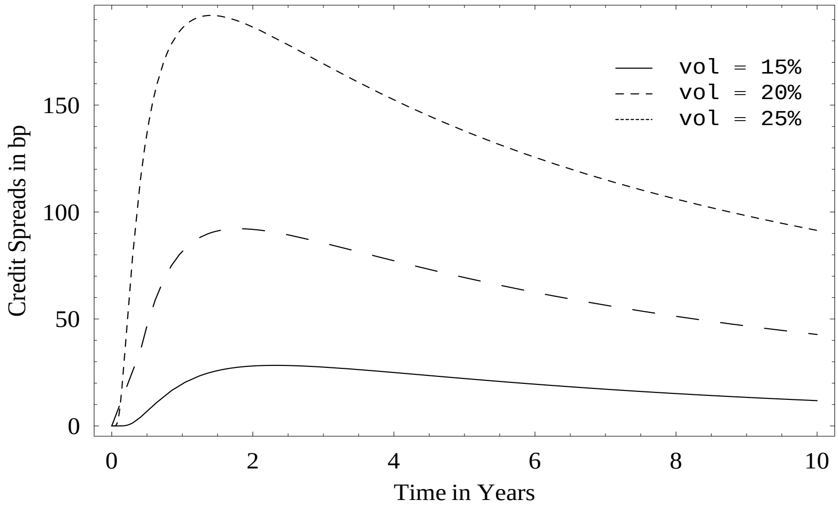

Figure 2: Term structure of credit spreads, varying asset volatility σ, in the classical approach.

asset dynamics (2), default probabilities are calculated as

$$p(T) = P[M_T < D] = P[\min_{s \leq T}(ms + \sigma W_s) < \log(D/V_0)].$$

where M is the historical low of firm values,

$$M_t = \min_{s \leq t} V_s.$$

Since the distribution of the historical low of an arithmetic Brownian motion is inverse Gaussian,[1] we have

$$p(T) = \Phi\left(\frac{\log(D/V_0) - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} \Phi\left(\frac{\log(D/V_0) + mT}{\sigma\sqrt{T}}\right). \quad (6)$$

We check whether this default definition is consistent with the payoff to investors. We need to consider two scenarios. The first is when $D \geq K$. If the firm value never falls below the barrier D over the term of the bond ($M_T > D$), then bond investors receive the face value $K < V_0$ and the equity holders receive the remaining $V_T - K$. However, if the firm value falls below the barrier

[1] To find that distribution, one first calculates the joint distribution of the pair $(W_t, \min_{s \leq t} W_s)$ by the reflection principle. Girsanov's theorem is used to extend to the case of Brownian motion with drift.

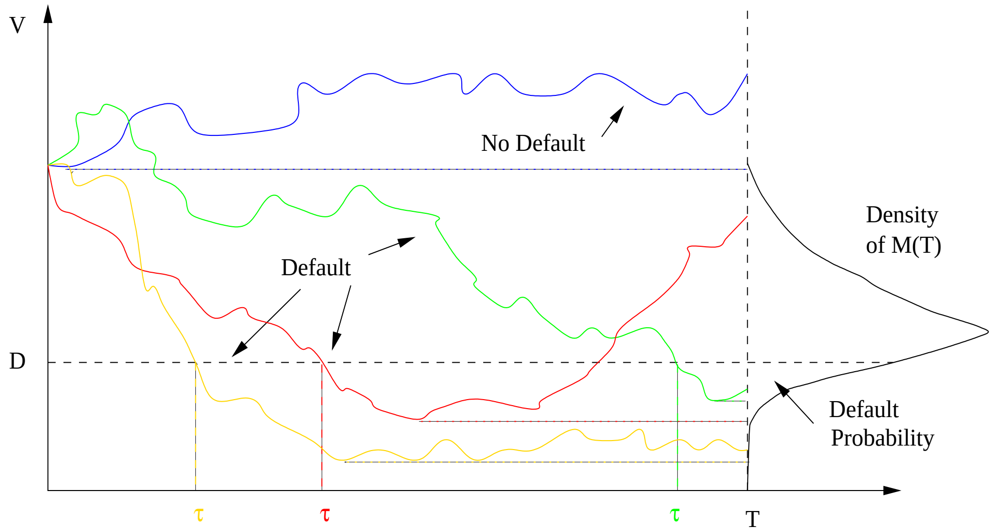

Figure 3: Default in the first-passage approach.

at some point during the bond's term ($M_T \leq D$), then the firm defaults. In this case the firm stops operating, bond investors take over its assets D and equity investors receive nothing. Bond investors are fully protected: they receive at least the face value K upon default and the bond is not subject to default risk any more.

This anomaly does not occur if we assume $D < K$ so that bond holders are both exposed to some default risk and compensated for bearing that risk. If $M_T > D$ and $V_T \geq K$, then bond investors receive the face value K and the equity holders receive the remaining $V_T - K$. If $M_T > D$ but $V_T < K$, then the firm defaults, since the remaining assets are not sufficient to pay off the debt in full. Bond investors collect the remaining assets V_T and equity becomes worthless. If $M_T \leq D$, then the firm defaults as well. Bond investors receive $D < K$ at default and equity becomes worthless.

Reisz & Perlich (2004) point out that if the barrier is below the bond's face value, then our earlier definition (5) does not reflect economic reality anymore: it does not capture the situation when the firm is in default because $V_T < K$ although $M_T > D$. We discuss two ways to avoid this inconsistency. The first is to re-define default as firm value falling below the barrier $D < K$ at any time before maturity *or* firm value falling below face value K at maturity. Formally, the default time is now given by

$$\tau = \min(\tau^1, \tau^2), \tag{7}$$

where τ^1 is the first passage time of assets to the barrier D and τ^2 is the maturity time T if assets $V_T < K$ at T and ∞ otherwise. In other words, the default time is defined as the minimum of the first-passage default time (5) and Merton's default time (1). This definition of default is consistent with the payoff to equity and bonds. Even if the firm value does not fall below the barrier, if assets are below the bond's face value at maturity the firm defaults. We get for the corresponding default probabilities

$$\begin{aligned} p(T) &= 1 - P[\min(\tau^1, \tau^2) > T] \\ &= 1 - P[\tau^1 > T, \tau^2 > T] \\ &= 1 - P[M_T > D, V_T > K] \\ &= 1 - P[\min_{t \leq T}(mt + \sigma W_t) > \log(D/V_0), mT + \sigma W_T > \log L] \end{aligned}$$

Using the joint distribution of an arithmetic Brownian and its running minimum, we get immediately

$$p(T) = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} \Phi\left(\frac{\log(D^2/(KV_0)) + mT}{\sigma\sqrt{T}}\right). \tag{8}$$

The corresponding payoff to equity investors at maturity is

$$E_T = \max(0, V_T - K)1_{\{M_T \geq D\}} \tag{9}$$

where 1_A is the indicator function of the event A. The equity position is equivalent to a European down-and-out call option position on firm assets V with strike K, barrier $D < K$, and maturity T. Pricing equity reduces to pricing European barrier options. In the Black-Scholes setting with constant interest rates and asset dynamics (2), we find the value

$$\begin{aligned} E_0 = V_0 &\left[\Phi\left(\frac{\bar{\nu}T - \log L}{\sigma\sqrt{T}}\right) - \left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}+1} \Phi\left(\frac{\bar{\nu}T + \log(D^2/(KV_0))}{\sigma\sqrt{T}}\right)\right] \\ &- Ke^{-rT}\left[\Phi\left(\frac{\nu T - \log L}{\sigma\sqrt{T}}\right) - \left(\frac{D}{V_0}\right)^{\frac{2r}{\sigma^2}-1} \Phi\left(\frac{\nu T + \log(D^2/(KV_0))}{\sigma\sqrt{T}}\right)\right] \end{aligned}$$

where $\nu = r - \sigma^2/2$ and $\bar{\nu} = r + \sigma^2/2$. The value of the bond is given as the residual value $V - E$.

The second way to avoid the inconsistency discussed above is to introduce a time-varying default barrier $D(t) \leq K$ for all $t \leq T$. For some constant $k > 0$, consider the deterministic function

$$D(t) = Ke^{-k(T-t)} \tag{10}$$

which can be thought of as the face value of the debt, discounted back to time t at a continuously compounding rate k. The firm defaults at

$$\tau = \inf\{t > 0 : V_t < D(t)\}. \tag{11}$$

Observing that

$$\{V_t < D(t)\} = \{(m-k)t + \sigma W_t < \log L - kT\}$$

we have for the default probability

$$p(T) = P[\min_{t \leq T}((m-k)t + \sigma W_t) < \log L - kT].$$

Now we have reduced the problem to calculating the distribution of the historical low of an arithmetic Brownian motion with drift $m - k$. We get

$$p(T) = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(Le^{-kT}\right)^{\frac{2}{\sigma^2}(m-k)} \Phi\left(\frac{\log L + (m-2k)T}{\sigma\sqrt{T}}\right). \tag{12}$$

The equity position is a European down-and-out call option on firm assets V with strike K, time-varying barrier $D(t)$, and maturity T. Merton (1973) gives a closed-form expression for the equity value. The position of the bond investors is defined through the corresponding payoffs. At default investors receive $D(\tau) \leq K$. In case the discount factor k is set equal to the risk-free rate r, bond investors receive an equivalent but default-free zero bond. This is sometimes called "equivalent recovery." If the firm does not default, they receive the full face value K at maturity. Bond values are given by the residual $V - E$.

The simple capital structure underlying our calculations so far is unrealistic. Pricing of individual bond issues can be performed under the assumption that firm default implies default on all outstanding debt. Suppose the firm has issued, among other debt, a zero coupon bond paying 1 at T if there is no default and R at T if the firm defaults by T. Here $R \in [0, 1]$ specifies the

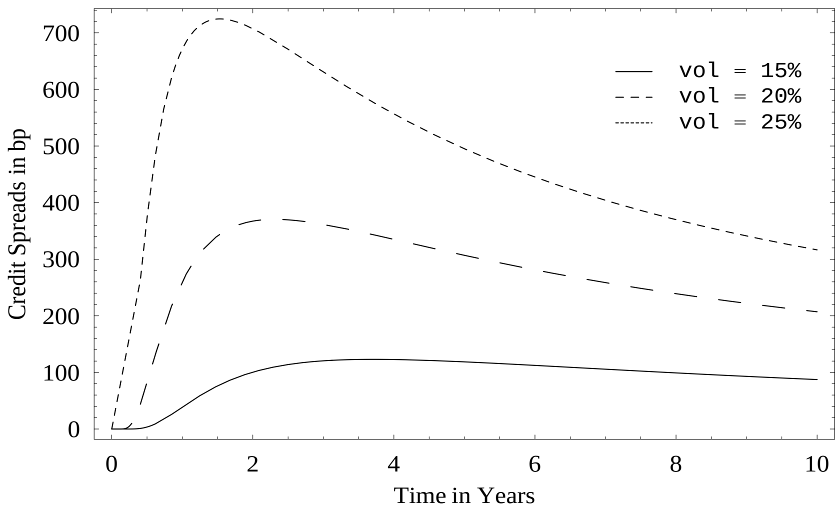

Figure 4: Term structure of credit spreads, varying asset volatility σ in the first-passage model.

recovery. In the absence of arbitrage, there is a market-implied probability Q such that the price of the bond is given as the expected discounted payoff under Q. We get for the bond price

$$e^{-rT} E^Q[R 1_{\{\tau \leq T\}} + 1_{\{\tau > T\}}] = e^{-rT} - e^{-rT}(1-R)q(T) \tag{13}$$

where $q(T)$ is the market-implied default probability. This probability can be calculated using the formulas (8) and (12) for $p(T)$ by setting $\mu = r$, corresponding to the asset dynamics under the market-implied probability Q. See Section 2.4 below for more details. Formula (13) says that the bond price is given by the price the bond would have if it were default risk free minus the present value of the default losses. From (4) we obtain for the credit spread associated with the bond

$$S(0,T) = -\frac{1}{T} \log\big(1 - (1-R)q(T)\big), \quad T > 0.$$

Consider a first passage model with time-varying default barrier $D(t)$ given by (10). Letting $R = 50\%$, $L = 60\%$ and $r = k = 6\%$, in Figure 4 we plot the term structure of credit spreads for varying asset volatilities. With increasing maturity T, the spread asymptotically approaches zero. This is at odds with empirical observation: spreads tend to increase with increasing maturity, reflecting the fact that uncertainty is greater in the distant future than in the near term. This discrepancy follows from two model properties: the firm value grows at a positive (riskfree) rate and the capital structure is constant. We can address this issue by assuming that the total debt grows at a positive rate, or that firms maintain some target leverage ratio as in Collin-Dufresne & Goldstein (2001).

2.3 Dependent Defaults

Credit spreads of different issuers are correlated through time. Two patterns are found in time series of spreads. The first is that spreads vary smoothly with general macro-economic factors in a correlated fashion. This means that firms share a common dependence on the economic environment, which results in *cyclical correlation* between defaults. The second relates to the jumps in spreads: we observe that these are often common to several firms or even entire markets. This suggests that the sudden large variation in the credit risk of one issuer, which causes a spread jump in the first place, can propagate to other issuers as well. The rationale is that economic distress is *contagious* and propagates from firm to firm. A typical channel for these effects are borrowing and lending chains. Here the financial health of a firm also depends on the status of other firms as well.

We want to incorporate these two default correlation mechanisms into the structural approach to credit. To introduce cyclical correlation, it is natural to assume that firm values of several firms are correlated through time. This corresponds to common factors driving asset returns. We consider the simplest case with two firms and asset dynamics

$$\frac{dV_t^i}{V_t^i} = \mu_i dt + \sigma_i dW_t^i, \quad V_0^i > 0, \quad i = 1, 2,$$

where $\mu_i \in \mathbb{R}$ is a drift parameter, $\sigma_i > 0$ is a volatility parameter, and (W^1, W^2) is a two-dimensional Brownian motion with correlation ρ. That is, Cov $(W_t^i, W_t^j) = \rho t$.

In the classical approach, we then obtain for the joint probability of firm 1 to default at T_1 (the fixed debt maturity) and firm 2 to default at T_2

$$p(T_1, T_2) = \Phi_2\left(\rho, \frac{\log(K_1/V_0^1) - m_1 T_1}{\sigma_1\sqrt{T_1}}, \frac{\log(K_2/V_0^2) - m_2 T_2}{\sigma_2\sqrt{T_2}}\right) \tag{14}$$

where $\Phi_2(\rho, \cdot, \cdot)$ is the bivariate standard normal distribution function with correlation ρ and $m_i = \mu_i - \frac{1}{2}\sigma_i^2$. In the first-passage approach we get for the joint probability of firm 1 to default before T_1 and firm 2 to default before T_2

$$p(T_1, T_2) = \Psi_2\big(\rho; T_1, T_2; \log(D_1/V_0^1), \log(D_2/V_0^2)\big)$$

where D_i is the constant default barrier of firm i and, holding $x, y \leq 0$ fixed, $\Psi_2(\rho; \cdot, \cdot; x, y)$ is the bivariate inverse Gaussian distribution function with correlation ρ. This function is given in closed-form in Iyengar (1985) and Zhou (2001*a*).

The joint default probability p provides a comprehensive characterization of the default risk of both firms. It describes simultaneously the individual likelihood of a firm to default and the likelihood that both firms default jointly. In the portfolio context we are often interested in the component of p describing the default dependence structure only. It turns out that we can isolate this dependence structure from p by means of a *copula*. Formally, the copula C of the default times (τ_1, τ_2) is a function that maps the individual default probabilities p_i into the joint default probability p,

$$p(T_1, T_2) = C\big(p_1(T_1), p_2(T_2)\big).$$

There is only one such mapping C if p is continuous.[2] In this case we can also go the other way around, and find C from a given p through

$$C(u, v) = p\big(p_1^{-1}(u), p_2^{-1}(v)\big)$$

for all u and v in $[0, 1]$. Here p_i^{-1} is the (generalized) inverse of the individual default probability. Accordingly, in the classical approach the default dependence structure can be represented through the Normal copula. In the first passage approach, the default dependence structure is given by the inverse Gaussian copula.

The default copula measures the complete non-linear dependence between the defaults. It satisfies the Fréchet bound inequality

$$\max(u + v - 1, 0) \leq C(u, v) \leq \min(u, v)$$

for all u and v in $[0, 1]$, since it is a joint distribution function with standard uniform marginals. If C takes on the lower bound, defaults are perfectly negatively correlated. This corresponds to an asset correlation of $\rho = -1$ and means that the time of default of one firm is a decreasing function of the default time

[2] A complete account of copulas can be found in Nelsen (1999).

of the other firm. If C takes on the upper bound, defaults are perfectly positively correlated. This corresponds to an asset correlation of $\rho = 1$ and means that one default time is an increasing function of the other. In case $p_1 = p_2$, both firms default literally at the same time. Finally, it is easy to check that in case $C(u, v) = uv$ defaults must be independent.

The function-valued copula C seems however not to be the most convenient measure for non-trivial default dependence. A bivariate scalar-valued measure is often more intuitive. One such measure is Spearman's rank correlation, cf. Embrechts, McNeil & Straumann (2001). For the default times τ_1 and τ_2 it is easily constructed as the linear correlation of the copula C, i.e.

$$\rho^\tau = 12 \int_0^1 \int_0^1 C(u, v) du dv - 3.$$

Rank default correlation ρ^τ is a function of the copula C only. It describes the degree of *monotonic* default dependence through a number in $[-1, 1]$, with the left (right) endpoint referring to perfect negative (positive) default dependence. Rank correlation should be contrasted with linear correlation of the default times τ_i and linear correlation of the Bernoulli default indicators $1_{\{\tau_i \leq T\}}$. These measures are often used in the literature; they describe the degree of *linear* default dependence through a number in $[-1, 1]$. Unless the default times/default indicators are jointly elliptically distributed, linear correlation based measures will misrepresent default dependence: they do not cover the non-linear part of the dependence. Rank correlation ρ^τ does not suffer from this defect: it summarizes monotonic dependence.

Asset correlation captures the dependence of firms on common economic factors in a natural way. Modeling default contagion effects is much more difficult. A straightforward idea is to consider a jump-diffusion model for firm value. We would stipulate that a downward jump in the value of a given firm triggers subsequent jumps in the firm values of other firms with some probability. This would correspond to the propagation of economic distress. This approach fails however due to the lack of (closed-form) results on the joint distribution of firms' historical asset lows. This is what we need to calculate the probability of joint default.

A more successful attempt is to introduce interaction effects through the default barriers D_i. Giesecke & Weber (2004) suppose the barrier is random and depends on the firm's liquidity state, which in turn depends on the default status of the firm's counterparties. If a firm's liquidity reserves are stressed due to a payment default of a counterparty, it finances the loss by issuing more

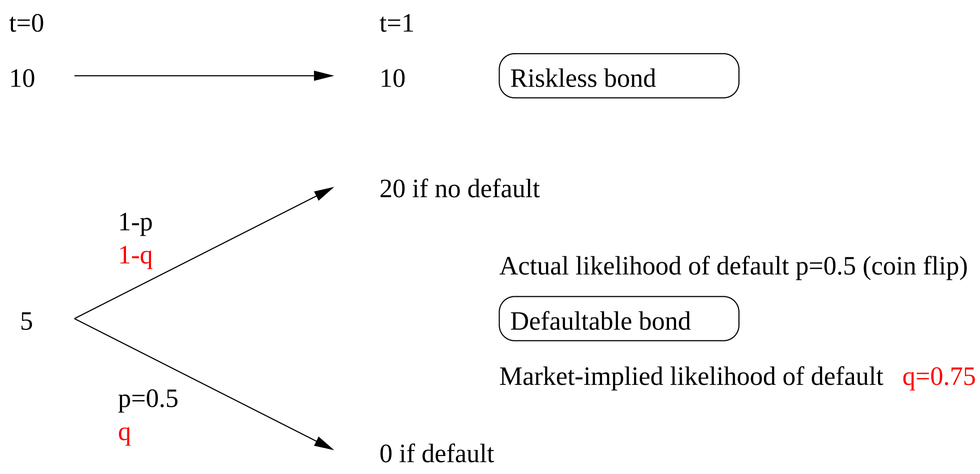

Figure 5: Actual vs. market-implied likelihood of default.

debt. This increases the default barrier: the firm is now more likely to default, all else being equal. With no counterparty defaults the default barrier remains unaffected. This model allows a closed-form approximation of the credit portfolio loss distribution.

2.4 Credit premium

Issuers of credit sensitive securities share a common dependence on the economic environment. It follows that aggregated credit risk cannot be diversified away. This undiversifiable or systematic risk commands a premium, which compensates risk-averse investors for assuming credit risk.

The credit premium is empirically well-documented and theoretically complex. Its importance relates to the uses of a quantitative credit model. As a default probability forecasting tool, a credit model must reflect the historical default experience. As a tool for pricing credit sensitive securities, it must fit observed market prices. To make use of both market data and historical default data in the calibration and application of a credit model, we need to understand the relationship between actual defaults and defaultable security prices. Here the risk premium comes into play: it maps the actual likelihood of default $p(T)$ into the market-implied likelihood of default $q(T)$ that is embedded in security prices, see (13).

We examine the difference between the two using a simple example, see Figure 5. We consider a one-period market with two securities, a riskfree bond paying 10 and trading at 10 (riskfree rates are zero) and a defaultable bond trading at 5, that pays 20 in case of no default and zero in case the issuer defaults by the end of the trading period. Suppose the actual probability of default is $p = 0.5$. This is however not the probability the market uses for pricing the bond: it would lead to a price of $p{\cdot}20 = 10$, which is double the price the bonds is actually trading. At this price, risk-averse investors would rather put their money into the riskfree bond that costs 10 as well, unless they get a discount as compensation for the default risk. The market requires a discount of 5, and the corresponding price reflects the market-implied probability of default q, which satisfies $5 = (1-q)20$. This yields $q = 0.75$, which is bigger than the actual probability of default $p = 0.5$. To account for risk aversion in calculating the expected payoff of the defaultable bond, the market puts more weight on unfavorable states of the world in which the firm defaults.

In the structural credit models with asset dynamics (2) and constant riskfree rates, the situation is only a little more complicated. Here the relationship between actual probabilities of events P and market-implied probabilities Q is well understood. For a fixed finite $T > 0$, it is characterized through a random variable Z_T, called the Radon-Nikodym density. In the absence of arbitrage opportunities, the density is uniquely determined through market prices of credit sensitive securities such as equity or debt as

$$Z_T = \exp\left(-\alpha W_T - \frac{1}{2}\alpha^2 T\right) \tag{15}$$

where W is the Brownian motion driving the uncertainty about firm assets and the constant α is the risk premium for this uncertainty. It is given as the excess return on firm assets over the riskfree return per unit of firm risk, measured in terms of asset volatility:

$$\alpha = \frac{\mu - r}{\sigma}. \tag{16}$$

If the market is risk averse, then α is positive: investors in credit-risky firm assets require a return that is higher than the riskfree return. The excess return on any credit sensitive security is given by its volatility times α.

Girsanov's theorem implies that the process defined by $W_t^Q = W_t + \alpha t$ is a Brownian motion under the market-implied probability Q. Hence the firm value grows at the riskfree rate r under Q; its dynamics are given by

$$\frac{dV_t}{V_t} = r dt + \sigma dW_t^Q, \quad V_0 > 0. \tag{17}$$

Above we obtained the actual default probability p when the firm grows at a rate μ. Setting $\mu = r$ in the formulas for $p(T)$ yields the market-implied default probability $q(T) = E^Q[1_{\{\tau \leq T\}}] = Q[\tau \leq T]$. Prices of securities are given through their expected discounted payoff under the probability Q.

2.5 Calibration

The calibration of a quantitative credit model is closely related to its use. To price single-name credit sensitive securities using a structural model, we need to calibrate the following vector of constant parameters:

$$(r, \sigma, V_0, K, D, T),$$

The first three parameters refer to firm value dynamics, whereas the remaining parameters relate to the debt of the firm. The barrier D is relevant only in the first passage approach. To use the model to forecast actual default probabilities, we need to calibrate additionally the growth rate μ of firm assets or, equivalently, the risk premium α. In a multiple firm setting we need to estimate asset correlations in addition to the single-name parameters.

Firm values are not directly observable. The goal is to estimate the parameters of the firm value process based on equity prices, which can be observed for public firms. Riskfree interest rates can be estimated from default-free Treasury bond prices via standard procedures. We bypass estimation of face value and maturity of firm debt from balance sheet data, which is non-trivial given the complex capital structure of firms. In practice these parameters are often fixed ad-hoc, as some average of short-term and long-term debt, for example. We introduce a more reasonable solution to this problem later.

We consider the classical approach. Given equity prices E_t and equity volatility σ_E, Jones, Mason & Rosenfeld (1984) and many others suggest to back out V_t and σ by numerically solving a system of two equations. The first equation relates the equity price to asset value, time and asset volatility:

$$E_t = f(V_t, t) \tag{18}$$

where $f(x, t)$ is the Black-Scholes pricing function for a European call with strike K and maturity T. The second equation relates the equity price to asset and equity volatility, the Delta of equity, and asset value:

$$E_t = \frac{\sigma}{\sigma^E} f_x(V_t, t) V_t, \tag{19}$$

where a subscript on f refers to a partial derivative. This relation is obtained from applying Itô's formula to (18), yielding

$$\begin{aligned} &df(V_t,t) \\ &= \left(f_x(V_t,t)\mu V_t + \frac{1}{2}f_{xx}(V_t,t)\sigma^2 V_t^2 + f_t(V_t,t)\right)dt + \sigma f_x(V_t,t)V_t dW_t, \end{aligned} \tag{20}$$

and comparing the diffusion coefficient to that of the equity value dynamics

$$dE_t = \mu^E E_t dt + \sigma^E E_t dW_t.$$

The constant μ^E is the equity growth rate.

We can use (18) and (19) to "translate" a time series of equity values into a time series of asset values and volatilities. As for the equity volatility, we can use the empirical standard deviation of equity returns, or implied volatilities from options on the stock. Given a time series of asset returns, the empirical growth rate yields an estimate of μ and hence the market price of credit risk (16). Further, given asset return time series of several firms, asset correlation can be estimated. Alternatively, we can introduce a linear factor model for normally distributed asset returns, which expresses the idea that firms share a common dependence on general economic factors:

$$\log\left(\frac{V_T^i}{V_0^i}\right) = w_i\psi_i + \epsilon_i.$$

Here ψ_i is a normally distributed systematic factor, which can be constructed as a weighted sum of global, country and industry specific factors. The constant w_i is the factor loading, and expresses the linear correlation between asset returns and the systematic factors. The ϵ_i are (mutually) independent normally distributed factors, which capture idiosyncratic risk in asset returns. The asset correlation is now determined through the factors loadings, see Crouhy, Galai & Mark (2000) for details.

This calibration procedure is not entirely satisfying. First, equation (19) is redundant: it was already used in deriving the equity pricing function (18). Second, equity volatility is typically estimated as the empirical standard deviation of equity returns, although (20) clearly shows that it is random, depending on asset value and time. This restriction is in fact necessary to obtain a unique solution to the system (18) and (19). Third, the distributions of the estimators are not readily available, making it hard to judge their statistical quality.

Fourth, the standard estimate of the firm growth rate is very poor: it is based on two asset return observations only.

An alternative maximum likelihood estimation procedure is based on the framework proposed by Duan (1994). Here we consider equity price data as transformed asset price data, with the equity pricing function defining the transformation. Suppose we observe a time series of daily equity values $(E^i)_{i=1,2,\ldots,n}$; we drop the time index for clarity. Letting $\Delta t = 1/240$, the likelihood function can be derived as

$$\mathcal{L}(E^1, \ldots, E^n) = \prod_{i=1}^{n-1} \frac{1}{\sqrt{2\pi\Delta t}\sigma V^{i+1}|f_x(V^{i+1}, 0, \sigma)|} e^{-\frac{1}{2\sigma^2\Delta t}\left[\log\left(\frac{V^{i+1}}{V^i}\right)-m\Delta t\right]^2}$$

where V^i is given as the solution of the equation

$$E^i = f(V^i, 0, \sigma), \tag{21}$$

if a unique solution exists. Here f is the equity pricing function with asset value, time and asset volatility as arguments. It is given through the underlying structural credit approach. While a unique solution to (21) exists in the classical approach, it is unclear to us whether such a solution exists in the first passage approach. The estimates $\hat{m}$ and $\hat{\sigma}$ are the parameter values that maximize $\log \mathcal{L}$ given the equity time series. Note that we obtain an estimate of the firm growth rate, enabling us to obtain an estimate of the market price of credit risk via (16). Given $\hat{\sigma}$, estimates of asset values $\hat{V}^i$ are obtained as the solutions to $E^i = f(\hat{V}^i, 0, \hat{\sigma})$, if such solutions uniquely exist. Duan (1994) shows that these estimators are asymptotically normal. Duan, Gauthier, Simonato & Zaanoun (2003) extend this procedure to a setting with multiple firms, and obtain estimates of asset correlations as well.

2.6 Can we predict the future?

To a certain extent, users of structural models implicitly assume they can. In structural models, firm value is the single source of uncertainty that drives credit risk. Investors observe the distance of default as it evolves over time. If the firm value has no jumps, this implies that the default event is not a total surprise. There are "pre-default events" which announce the default of a firm. In the first passage approach, we can think of a pre-default event as the first time assets fall dangerously close to the default barrier, see Figure 6. Mathematically, there is an increasing sequence of event times $(\tau(n))$ that converge to the default time τ; we say the default is *predictable*.

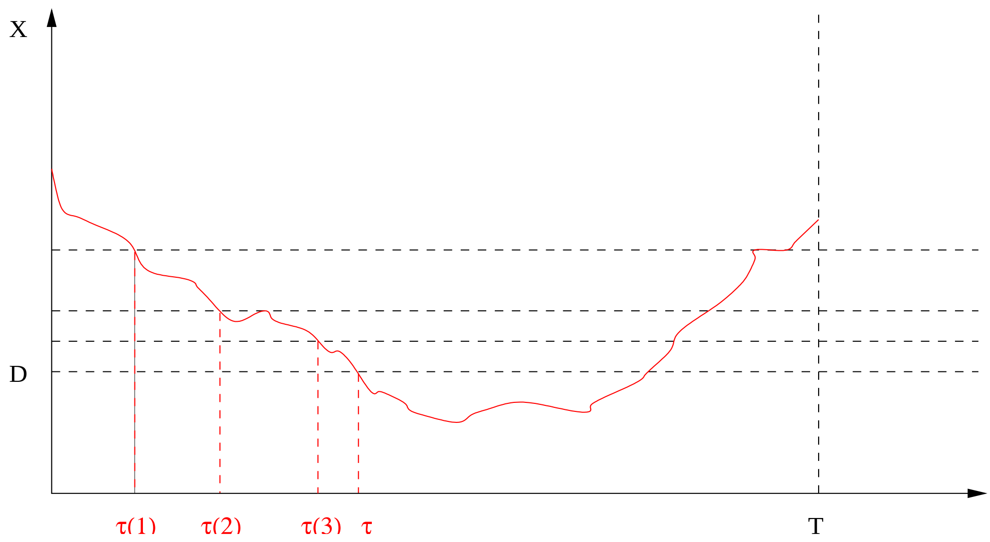

Figure 6: Announcing the default τ by a sequence $(\tau(n))$ of "pre-defaults" in the first-passage approach.

This predictability of default is not just a technical obscurity, but has significant implications for the fitting of structural models to market prices. First, since default can be anticipated, the model price of a credit sensitive security converges continuously to its recovery value. Second, the model credit spread tends to zero with time to maturity going to zero:

$$\lim_{T \downarrow t} S(t, T) = 0 \tag{22}$$

almost surely, see Giesecke (2001*b*). Quite telling in this regard are the credit spreads implied by the classical and first-passage approaches, see Figures 2 and 4. Both properties are at odds with intuition and market reality. Market prices do exhibit surprise downward jumps upon default. Even for very short maturities in the range of weeks, market credit spreads remain positive. This indicates investors do have substantive short-term uncertainty about defaults, in contrast to the predictions of the structural models.

3 Reduced form credit models

Reduced form models go back to Artzner & Delbaen (1995), Jarrow & Turnbull (1995) and Duffie & Singleton (1999). Here we assume that default occurs without warning at an exogenous default rate, or intensity. The dynamics of the intensity are specified under the market-implied probability. Instead of asking why the firm defaults, the intensity model is calibrated from market prices.

3.1 Default intensity

In the structural approach, the dynamics of default are derived from the definition of default in terms of assets and liabilities. Lacking an economic default definition, we prescribe these dynamics exogenously, directly under the market-implied probability Q. The problem can be cast in the framework of point processes. Taking as given the random default time τ, we define the default process N by

$$N_t = 1_{\{\tau \leq t\}} = \begin{cases} 1 & \text{if} \quad \tau \leq t \\ 0 & \text{if} \quad \text{else.} \end{cases}$$

This is a point process with one jump of size one at default.

Since the default process is increasing, it has an upward trend: the conditional probability at time t that the firm defaults by time $s \geq t$ is as least as big as N_t itself. A process with this property is called a submartingale. A process with zero downward trend is called a martingale. This is a "fair" process in the sense that the expected gain or loss is zero.

The Doob-Meyer decomposition theorem enables us to isolate the upward trend from N. This fundamental result states that there exists an increasing process A^τ starting at zero such that $N - A^\tau$ becomes a martingale, see Dellacherie & Meyer (1982). The unique process A^τ counteracts the upward trend in N; it is therefore often called compensator.

Interestingly, the analytic properties of the compensator correspond to the probabilistic properties of default. For example, the compensator is continuous if and only if the default time τ is unpredictable. In this case the default comes without warning; a sequence of announcing pre-default times does not exist. This is a desirable model property as we shall see, since it allows to fit the model to market credit spreads.

The compensator describes the cumulative, conditional likelihood of default. In the reduced form approach to credit, the compensator is parameterized through a non-negative process λ by setting $A^\tau_t = A_{\min(t,\tau)}$ with

$$A_t = \int_0^t \lambda_s ds. \tag{23}$$

With this assumption, λ_t describes the conditional default rate, or *intensity*: for small Δt and $t < \tau$, the product $\lambda_t \cdot \Delta t$ approximates the market-implied probability that default occurs in the interval $[t, t+\Delta t)$. Any given non-negative process λ can be used to parameterize the dynamics of default. No economic model of firm default is needed for this purpose any more!

Example 3.1. Suppose λ is a constant. Then N is a homogeneous Poisson process with intensity λ, stopped at its first jump. Thus τ is exponentially distributed with parameter λ and the market-implied default probability is given by

$$q(T) = 1 - e^{-\lambda T}.$$

Given the default probability, we can calculate the intensity as

$$\lambda = \frac{d(T)}{1 - q(T)}$$

where d is the density of q. In view of this representation, in the statistics literature λ is often called hazard rate. □

Example 3.2. Suppose $\lambda = \lambda(t)$ is a deterministic function of time t. Then N is an inhomogeneous Poisson process with intensity function λ, stopped at its first jump. The default probability is given by

$$q(T) = 1 - e^{-\int_0^T \lambda(u) du}.$$

A simple but useful parametric intensity model is

$$\lambda(t) = h_i, \quad t \in [T_{i-1}, T_i), \quad i = 1, 2, \ldots \tag{24}$$

for constants h_i and T_i, which can be calibrated from market data. □

Example 3.3. Suppose that $\lambda = (\lambda_t)$ is a stochastic process such that conditional on the realization of the intensity, N is an inhomogeneous Poisson process stopped at its first jump. Then N is called a Cox process, or doubly-stochastic Poisson process. The conditional default probability given the intensity path up to time T is given by $1 - \exp(-\int_0^T \lambda_u du)$. By the law of iterated expectations we find the default probability

$$q(T) = 1 - E^Q[e^{-\int_0^T \lambda_u du}].$$

Lando (1998) introduces this framework to model default as the first time a continuous-time Markov credit rating chain U with state space $\{1, \ldots, Y\}$ hits the absorbing state Y. State 1 is interpreted as the highest credit rating category, state $Y-1$ is interpreted as the lowest rating before default, and state Y is the default state. The dynamics of U are described by a generator matrix Λ with transition intensities $\lambda_{i,j}(X_t)$, which are modeled as (continuous) functions of some state process X:

$$\Lambda_t = \begin{pmatrix} -\lambda_1(X_t) & \lambda_{1,2}(X_t) & \ldots & \lambda_{1,Y}(X_t) \\ \lambda_{2,1}(X_t) & -\lambda_2(X_t) & \ldots & \lambda_{2,Y}(X_t) \\ \vdots & & & \\ \lambda_{Y-1,1}(X_t) & -\lambda_{Y-1,2}(X_t) & \ldots & \lambda_{Y-1,Y}(X_t) \\ 0 & 0 & \ldots & 0 \end{pmatrix}$$

where

$$\lambda_i(X_t) = \sum_{j=1, j\neq i}^{Y} \lambda_{i,j}(X_t), \quad i = 1, \ldots, Y-1.$$

The state process "drives" the risk of rating transitions. For small Δt we can think of $\lambda_{i,j}(X_t)\Delta t$ as the probability that the firm currently in rating class i will migrate to class j within the time interval Δt. Consequently, $\lambda_i(X_t)\Delta t$ is the probability that there will be any rating change in Δt for a firm currently in class i. This generalizes Jarrow, Lando & Turnbull (1997), where the transition intensities $\lambda_{i,j}$ are assumed to be constant. With $\tau = \inf\{t \geq 0 : U_t = Y\}$, the process N is a Cox process with random default intensity $\lambda_t = \lambda_{U_t,Y}(X_t)$ at time t. Note that the default intensity is represented by the last column in the above generator matrix Λ. □

Example 3.4. Suppose that $\lambda = (\lambda_t)$ is a basic affine stochastic process with dynamics under the market-implied probability given by

$$d\lambda_t = \kappa(\bar{\lambda} - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + dJ_t. \tag{25}$$

Here, W is a standard Brownian motion which drives the continuous changes of the issuer's credit quality over time; the parameter σ controls the diffusive fluctuations. Abrupt and unexpected changes in the credit quality are modeled through the independent compound Poisson process J with jump arrival intensity j and jump size that is independent and exponentially distributed with mean ν. The parameter $\bar{\lambda}$ controls the long-run mean, and κ controls the speed of mean-reversion: we have $E^Q[\lambda_t] \to \bar{\lambda} + j\nu/\kappa$ as $t \to \infty$. Under technical conditions stated in Duffie, Schroder & Skiadas (1996), for general stochastic intensities the default probability is given by

$$q(T) = 1 - E^Q[e^{-\int_0^T \lambda_u du}]. \tag{26}$$

For the dynamics (25), this can be calculated in closed-form as

$$q(T) = 1 - e^{(a+b\lambda_0)T},$$

where a and b are explicitly given in Duffie & Garleanu (2001). A special case of (25) is the square-root diffusion model of Cox, Ingersoll & Ross (1985), where $j = 0$. Other tractable affine intensity models include the Ornstein-Uhlenbeck process of Vasicek (1977) and the class of jump diffusion processes discussed in Duffie, Pan & Singleton (2000). □

These examples constitute only a small sample of possible parameterizations of the default intensity. There are many more choices, often borrowed from the classical term structure models based on the short term interest rate. This is motivated by the close analogy of defaultable term structure models and classical, non-defaultable term structure models to which we turn next.

3.2 Valuation

The description of the default dynamics through the market-implied default intensity λ leads to tractable valuation formulas. Below, we describe several different specifications of these formulas corresponding to different units for the value recovered by investors at default. We refer to Duffie & Singleton (2003) for a clear, detailed discussion of this material.

We consider a zero coupon bond paying 1 at maturity T if there is no default and R at T if the firm defaults before time T. Here the variable $R \in [0, 1]$ specifies the *recovery of face value* on the bond. With constant interest rates and constant recovery, the bond price is

$$B_0^T = e^{-rT} E^Q[R 1_{\{\tau \le T\}} + 1_{\{\tau > T\}}] = e^{-rT} - e^{-rT}(1 - R)q(T) \tag{27}$$

where $q(T)$ is the market-implied default probability. As in (13), the value of the bond is the value of an otherwise equivalent riskfree bond minus the present value of the default loss. If the intensity is constant (Example 3.1) and recovery is zero, we obtain

$$B_0^T = e^{-rT}(1 - q(T)) = e^{-(r+\lambda)T}. \tag{28}$$

This means the value of the defaultable bond is calculated as if the bond were riskfree by using a default-adjusted discount rate. The new discount rate is the sum of the riskfree rate r and the intensity λ. This parallel between pricing formulas for defaultable bonds and otherwise equivalent default free bonds is one of the best features of reduced form models. As we discuss below, the parallel extends to more complicated securities.

A second specification of recovery at default is called *equivalent recovery.* Here bond investors receive at τ a fraction $R \in [0, 1]$ of an equivalent but default-free bond. Assuming constant recovery the bond price is

$$\begin{aligned} B_0^T &= E^Q[e^{-r\tau} R e^{-r(T-\tau)} 1_{\{\tau \leq T\}} + e^{-rT} 1_{\{\tau > T\}}] \\ &= e^{-rT} E^Q[(1-R) 1_{\{\tau > T\}} + R] \\ &= e^{-rT}(1-R)(1-q(T)) + e^{-rT} R \end{aligned} \tag{29}$$

which is the value of $1 - R$ zero recovery defaultable bonds plus the value of R riskfree zero bonds. It is easy to see that (29) is equal to (27).

In the *fractional recovery* scheme investors receive at τ a fraction $R \in [0, 1]$ of the bond's market value just before default. Mathematically, this value is $B_{\tau-}^T = \lim_{t \uparrow \tau} B_t^T$. In this setup, the bond price is

$$B_0^T = E^Q[e^{-r\tau} R B_{\tau-}^T 1_{\{\tau \leq T\}} + e^{-rT} 1_{\{\tau > T\}}]. \tag{30}$$

If the recovery and intensity are constant,

$$B_0^T = e^{-(r+(1-R)\lambda)T}. \tag{31}$$

This is the value of a zero recovery defaultable bond when the issuer's default intensity is "thinned" to $\lambda(1 - R)$. The intuition behind (31) is as follows. Suppose the bond defaults with intensity λ. At default, the bond becomes worthless with probability $(1 - R)$, and its value remains unchanged with probability R. Clearly, the pre-default value $B_{\tau-}^T$ of the bond is not changed by this way of looking at default. Consequently, for pricing we can ignore the "harmless" default, which occurs with intensity λR. We then price the bond as if it had zero recovery and a default intensity $\lambda(1-R)$. Formula (31) is then implied by (28).

The results for the valuation of more complex credit sensitive securities are analogous. We consider a credit sensitive security specified by the triple (T, c_T, R). It pays the amount c_T at T if no default occurs before T, the maturity of the security. In case of default, investors receive a (random) fraction of the security's pre-default value. It is modeled with a stochastic process R. If default occurs at time τ, the recovery fraction is $R_\tau \in [0,1]$. Under technical conditions stated in Duffie & Singleton (1999), the security price is given by the convenient formula

$$e^{-rT} E^Q \left[c_T e^{-\int_0^T (1-R_s)\lambda_s ds} \right]. \tag{32}$$

Also in the general case a credit sensitive security can be valued as if it were not sensitive to credit risk by using an adjusted rate for discounting payoffs.

We take a closer look at the credit spreads implied by reduced form models. In case recovery is zero and some technical conditions are satisfied, we can show that

$$\lim_{T \downarrow t} S(t,T) = \lambda_t$$

almost surely. This should be contrasted with the structural models, where the spread goes to zero with time to maturity going to zero, see (22). In the reduced form models the default event is unpredictable, it comes without warning. There is always short-term uncertainty about the default event, for which investors demand a premium. This premium, expressed in terms of yield, is given by the intensity.

The unpredictability of default has another important consequence. In line with empirical observation, the model price of a credit sensitive security will abruptly drop to its recovery value upon default. This is in direct conflict with the structural models considered above in which the price converges to its default contingent value.

3.3 Default Correlation

In the reduced form model we can introduce cyclical default correlation by assuming that firms' default intensities are smoothly correlated through time. An effective framework for this is the Cox process model of Example 3.3, extended to the multivariate case. Suppose that the indicator processes of the default times $\tau_1, \ldots, \tau_n$ with respective intensities $\lambda^1, \ldots, \lambda^n$ form a multivariate Cox process "driven" by some state process X. The state process includes the systematic, economy-wide and idiosyncratic factors driving the credit risk of firms. Conditional on X, firm defaults are independent. Joint survival probabilities can be calculated by observing that the first default time $\tau = \min \tau_i$

has intensity $\lambda^1 + \ldots + \lambda^n$ if $\tau_i \neq \tau_j$ almost surely for $i \neq j$, which is satisfied in the Cox process framework. Let $0 < T_1 \leq \ldots \leq T_n$ be given horizons. Then

$$Q[\tau_1 > T_1, \ldots, \tau_n > T_n] = E^Q[e^{-\int_0^{T_n} \sum_{i:T_i > s} \lambda_s^i ds}],$$

which is easily calculated in the basic affine setting of Example 3.4. Of course, standard arguments yield also the joint default probability q.

We can take advantage of the default copula C corresponding to q the same way we did within the structural models by calculating $C(u_1, \ldots, u_n) = q(q_1^{-1}(u_1), \ldots, q_n^{-1}(u_n))$, where q_i^{-1} is the (generalized) inverse of the individual market-implied default probability q_i of firm i. The specific copula will depend on the particular choice of the functional form of the λ^i. More generally, we can use any copula to build tractable models for correlated defaults. Together with arbitrary marginal default probabilities q_i, any copula C specifies a proper joint default probability q via

$$q(T_1, \ldots, T_n) = C(q_1(T_1), \ldots, q_n(T_n)).$$

We can use, for example, the Gaussian copula corresponding to the structural joint default probability (14) together with reduced form marginal default probabilities from Examples 3.1 through 3.4. This is a popular modeling choice in practice, since the asset correlations parameterizing the Gaussian copula are relatively easy to come by. It is somewhat inconsistent however, given the incompatible assumptions of the models underlying the copula and the marginals. Alternatively we may choose some parametric copula family and combine it with some marginal default probability model. This has some drawbacks as well. Due to the lack of data allowing to infer default correlation, calibration of the copula parameters is quite difficult. Most importantly, the choice of the copula is arbitrary–there is no "natural choice." This introduces a large amount of model risk, since different copulas lead to quite different joint default characteristics. In lack of empirical data of correlated defaults, it is hard to say which characteristics are natural. For a discussion of these model risk issues we refer to Frey & McNeil (2004).

Taking account of contagious default correlation is not an easy exercise. The idea is that there are correlated jumps in firms' default intensities, corresponding to the correlated jumps we observe in credit spreads. A variant of this assumes that there are market-wide events that can trigger joint defaults, see Duffie & Singleton (1998) and Giesecke (2003). Another variant assumes that the default intensity of a firm depends explicitly on the default status of related counterparty firms in the market. A parametrization of this idea is

$$\lambda_t^i = h_t^i + \sum_{j \neq i} a^j N_t^j$$

see Jarrow & Yu (2001). Here h^i is the base default intensity and N^j is the default indicator process of firm j. The parameters a^j are chosen such that λ^i is non-negative. To avoid running into a circularity problem, one can suppose that only the default of designated "primary" firms has an effect on other, "secondary" firms.

While Jarrow & Yu (2001) focus on the pricing of credit sensitive securities in the presence of contagion effects, it is difficult to calculate joint default probabilities and portfolio loss distributions within this approach. As Davis & Lo (2001) and Giesecke & Weber (2003) show, one can obtain tractable closed-form characterizations of loss distributions at the cost of more restricting assumptions, which relate to the homogeneity of firms and the symmetry in their counterparty relations.

3.4 Calibration

Reduced form models are typically formulated directly under the market-implied probability. This suggests that we calibrate directly from market prices of various credit sensitive securities. One often uses liquid debt prices or credit default swap spreads, although Jarrow (2001) argues that equity is a good candidate as well. Depending on the characteristics of the calibration security, it may be necessary to make parametric assumptions about the recovery process as well. With fractional recovery and zero bonds for example, the problem is to choose the parameters of the adjusted short rate model $r + (1 - R)\lambda$ such that model bond prices (32) best fit observed market prices. Here one can either parameterize the adjusted short rate directly or specify the component processes separately. With a separate specification identification problems may arise, since only the product $(1 - R)\lambda$ enters the pricing formula (32). In general, in the estimation problem one can draw from the experience related to non-defaultable term structure models, given the close analogy to reduced form defaultable models. We refer to Dai & Singleton (2003) for an overview of available techniques. Standard methods include maximum likelihood and least squares.

4 Incomplete information credit models

The incomplete information framework provides a common perspective on the structural and reduced form approaches to analyzing credit. This perspective enables us to see models of both types as members of a common family. This family contains previously unrecognized structural/reduced form hybrids, some of which incorporate the best features of both traditional approaches. Incomplete information credit models were introduced by Duffie & Lando (2001), Giesecke (2001*b*) and Çetin, Jarrow, Protter & Yildirim (2002). A non-technical discussion of incomplete information models is in Goldberg (2004).

4.1 Default trend

Underlying *all* credit models is the increasing default process N and its upward trend A. Thanks to the Doob-Meyer decomposition, the trend can be isolated from the default process. The difference is a martingale, a fair process whose expected gains or losses are zero. The trend represents the fair cumulative compensation for the short-term credit risk embedded in the default process. If there is short-term uncertainty about default in any state of the world, the trend can be used to estimate default probabilities and price credit sensitive securities.

In traditional structural models default can be anticipated. In this case there is no short-term credit risk that would require compensation. Correspondingly, the trend is trivially given by the default process itself. In reduced form models, it is assumed that default cannot be anticipated, so there is short-term credit risk by assumption. The non-trivial trend is directly parameterized through the intensity:

$$A_t = \int_0^t \lambda_s ds, \tag{33}$$

which defines the trend as the cumulative default intensity. In this situation the dynamics of model default probabilities and security prices are immediately implied by the exogenous intensity dynamics.

Instead of focusing on the default intensity and making ad-hoc assumptions about its dynamics, incomplete information models seek to specify the trend based on a model definition of default. Here we provide an endogenous characterization of the trend in terms of a firm's assets and liabilities via an underlying structural model. But this works only if we can modify the underlying structural model to admit short-term credit risk.

There are two approaches to introduce short-term uncertainty into structural models. The first is to allow for "surprise" jumps in the firm value, as in Zhou (2001*b*), Hilberink & Rogers (2002) and Kijima & Suzuki (2001). In this situation there is always a chance that the firm value jumps below the default barrier. This cannot be anticipated. However, there is also a chance that the firm just "diffuses" to the barrier, as in the traditional models with continuous value process. Here default can be anticipated. So depending on the state of the world, there may or may not be short-term credit risk.

There is another approach that guarantees default cannot be anticipated so there is short-term credit risk in any state of the world. This approach arises through a re-examination of the informational assumptions underlying the traditional structural models. In these models, it is implicitly assumed that the information we need to calibrate and run the model is publicly available. This information includes the firm value process and its parameters as well as the default barrier. In the incomplete information framework, we address the fact that in reality, our information about these quantities is imperfect. The information we have is much coarser than the idealized traditional structural models suggest, as highlighted by the high profile scandals at Enron, Tyco and WorldCom. Concretely this means that we may not be sure either of the true value of the firm or of the exact condition of the firm that will trigger default. It follows that we are always uncertain about the distance to default. Thus, default is a complete surprise: it cannot be anticipated. The non-trivial trend, which represents the compensation for the associated short-term credit risk, can always be characterized explicitly in terms of firm assets and default barrier. Here are two case studies.

Example 4.1 (I^2 credit model). Suppose default is described by the first passage model of Section 2.2. Assume that we cannot observe the default time-independent barrier D. Let D be independent with continuous distribution function G on $(0, V_0)$. Giesecke (2001*b*) shows that the trend A is given by

$$A_t = -\log G(M_t) \tag{34}$$

where, as above, M_t is the historical low of firm value at time t. In view of (33), we need only differentiate the trend to get the intensity. Under the assumption that G is differentiable, the derivative of A is however zero almost surely. This means that we cannot write the trend as in (33) in terms of an intensity. $\square$

Example 4.2. In the first passage model of Section 2.2, suppose we do not observe the firm value directly but instead receive noisy asset reports from time to time. Let $f(\cdot, t)$ be the conditional density of the log-firm value at time t on $(d, 0)$ where $d = \log(D/V_0)$. Duffie & Lando (2001) show that

$$A_t = \frac{1}{2}\sigma^2 \int_0^t f_x(d, s)ds$$

Here an intensity exists and is given by $\lambda_t = \frac{1}{2}\sigma^2 f_x(d, t)$. $\square$

The trend is the key to the calculation of default probabilities and prices of credit sensitive securities. Under technical conditions stated in Giesecke (2001*b*), we have the generalized reduced form formula

$$q(T) = 1 - E^Q[e^{-A_T}] \tag{35}$$

where A is the default trend under the market-implied probability Q. This formula simplifies to the reduced form formula (26) if the trend admits an intensity. There are closed form expressions for $q(T)$ in some cases.

Example 4.3 (I^2 credit model). Suppose the default barrier D is uniform on $(0, V_0)$. We have for the market-implied default probability

$$q(T) = 1 + \left(\frac{\sigma^2}{2r} - 1\right)\Phi\left(\frac{\nu\sqrt{T}}{\sigma}\right) - e^{rT}\left(1 + \frac{\sigma^2}{2r}\right)\Phi\left(\frac{-\bar{\nu}\sqrt{T}}{\sigma}\right) \tag{36}$$

where $\nu = r - \sigma^2/2$ and $\bar{\nu} = r + \sigma^2/2$. $\square$

We reconsider the fractional recovery credit sensitive security specified by the triple (T, c_T, R), that we introduced in Section 3.2. Under technical conditions stated in Giesecke & Goldberg (2003*b*), we have for the security's pre-default value the generalized reduced form formula

$$e^{-rT}E^Q\left[c_T e^{-A_T(R)}\right] \tag{37}$$

where $A_t(R) = \int_0^t (1 - R_s)dA_s$ is the fractional recovery trend. More precisely, it is the upward trend of the default process $(1 - R_\tau)N$ under fractional recovery. If the trend admits an intensity, (37) simplifies to (32).

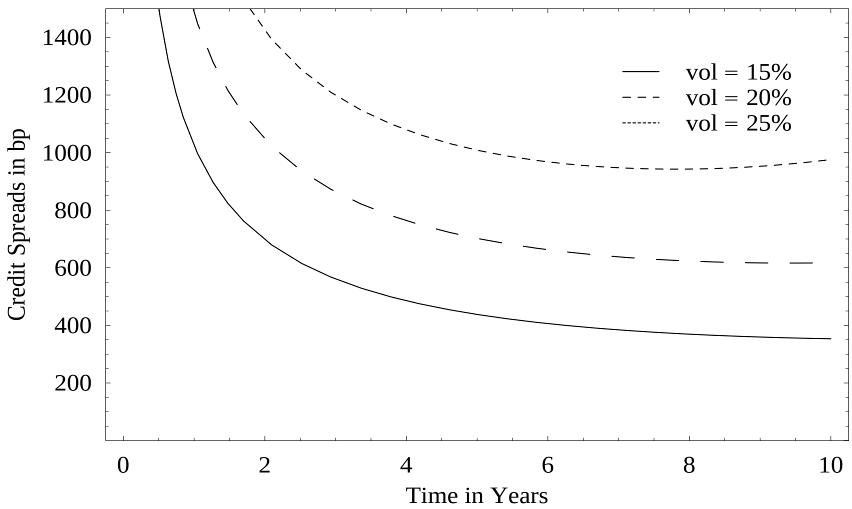

Figure 7: Term structure of credit spreads, varying asset volatility σ in the I^2 model.

The incomplete information models share many of the good properties of both structural and reduced form models while avoiding their difficulties. While built on an intuitive and economically meaningful structural approach, default cannot be anticipated as in the traditional structural models. This has several desirable consequences. First, any incomplete information model admits a non-trivial trend that can be characterized explicitly. The trend can be used to calculate default probabilities and prices of credit sensitive securities through tractable generalized reduced form formulas. In the traditional structural models these convenient reduced form formulas fail. Second, consistent with empirical observations, prices of credit sensitive securities drop abruptly to their recovery values upon default. Third, short-term credit spreads are typically bounded away from zero. To illustrate this, we consider a zero recovery zero bond with face value 1 maturing at T. The bond is priced at $e^{-rT}(1-q(T))$. Letting $r = 6\%$, in Figure 7, we plot the corresponding credit spreads

$$S(0,T) = -\frac{1}{T}\log\big(1-q(T)\big), \quad T > 0,$$

in the I^2 model. Giesecke & Goldberg (2003a) calibrate the I^2 model from market data and further analyze its empirical properties. In particular, the I^2 model output is empirically compared to a traditional first passage model. Two main conclusions can be drawn. The I^2 model reacts more quickly since it takes direct account of the entire history of public information rather than just current values. This can be seen from the structure of the trend in (34): it depends on the historical low of the firm value. Furthermore, the I^2 model predicts positive short spreads for firms in distress. The traditional first passage model always predicts that short spreads are zero.

4.2 Dependent defaults

Since incomplete information models are based on the structural approach, we can model cyclical default correlation through firm value correlation.

Contagious default correlation arises very naturally with incomplete information. Consider the I^2 model. As discussed in detail in Giesecke (2001*a*), with defaults of firms arriving over time, we learn about the unobserved default barriers of the surviving firms. This means we update the distribution we put on a firm's default barrier with the information we extract from the unanticipated defaults of counterparty firms, and re-assess firms' default probabilities. The situation in which we do not directly observe firm values (Example 4.2) is very similar; it is analyzed in Collin-Dufresne, Goldstein & Helwege (2002). In both scenarios the "contagious" jumps in credit spreads we observe in credit markets are implied by informational asymmetries.

The same way we introduced the trend in the single firm case to estimate default probabilities and prices of securities, we can develop the concept of the trend in a situation with multiple firms under incomplete information. The trend can then be used to estimate prices of securities that depend on the credit risk of multiple firms. It can also be used to construct efficient simulation algorithms for the simulation of correlated default events. This analysis is carried out in Giesecke (2002).

4.3 Credit premium

The credit risk premium is the mapping between the actual probability P and the market-implied probability Q. To understand the structure of the premium, we examine the dynamics of firm value and corporate liabilities in the I^2 model. We argued above that thanks to the unpredictability of default, prices of credit sensitive claims including firm equity and debt drop precipitously at default. Empirical observation shows that equity drops to near zero. This makes sense since equity holders have no stake in the firm after default. The value of the bonds is diminished by bankruptcy costs, which is described by some fractional

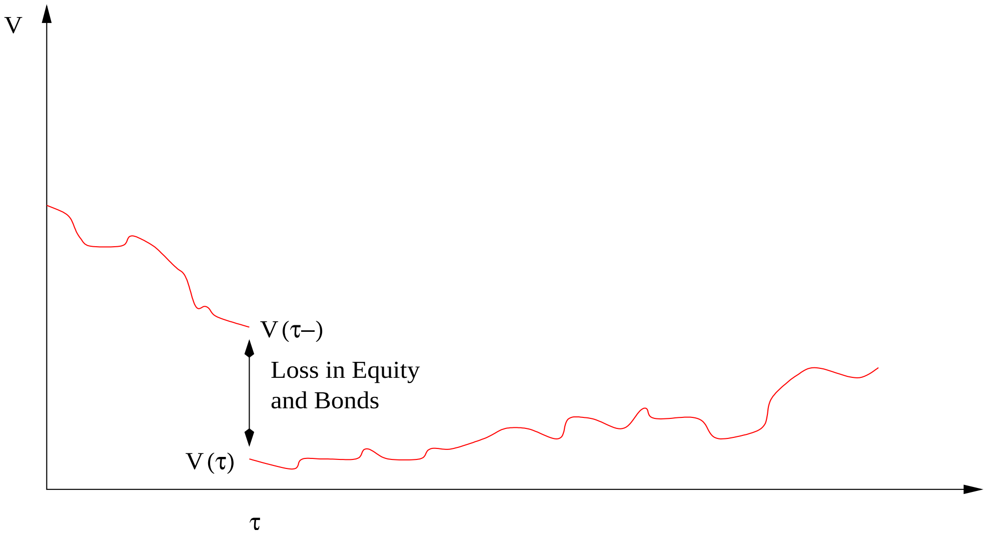

Figure 8: Firm value in the incomplete information model.

recovery process R. Consequently, firm value, which is equal to the sum of equity and debt values, also drops at default. This is shown in Figure 8. If default were to occur at time t, the combined default losses in equity and debt value relative to V are given by

$$J_t = \frac{1}{V_{t-}}\big(E_{t-} + (1 - R_t) \cdot B_{t-}\big).$$

Here E denotes the value of equity and B denotes the value of the bonds. If prior to default firm value follows a geometric Brownian motion, then the firm value process can hence be written as the jump diffusion

$$\frac{dV_t}{V_{t-}} = \mu dt + \sigma dW_t - J_t dN_t. \tag{38}$$

This shows that there are two sources of uncertainty related to firm value. The first is the diffusive uncertainty represented by the volatility σ. The second is the uncertainty associated with the downward jump in firm value at default.

The density describing the relation between the probabilities P and Q is now richer than (15) as Giesecke & Goldberg (2003*b*) show in the context of the I^2 model. As in the traditional structural models, the density is parameterized by the risk premium. In the incomplete information models the risk premium can be decomposed into to components, which correspond to the two sources of uncertainty. The diffusive risk premium α compensates investors for the diffusive uncertainty in firm value. As in the traditional structural models, it is realized as a change to the drift term in firm value dynamics: $\mu - r = \alpha_t \sigma$. The default event risk premium β is not present in the traditional structural models. It compensates investors for the jump uncertainty in firm value and is realized as a change to the default probability. Driessen (2002) empirically confirms that this event risk premium is a significant factor in corporate bond returns.

In Giesecke & Goldberg (2003*b*), it is shown that the assumption of no arbitrage is realized in the mathematical relationships among α, β, the recovery rate assumed by the market, and the coefficients of the price processes of traded securities. The price processes depend explicitly on the leverage ratio, so the premia α and β do as well. In this case the density depends on firm leverage. As Giesecke & Goldberg (2004*b*) discuss, this violates an important condition for the Modigliani & Miller (1958) theorem. The I^2 model is therefore not consistent with the Modigliani-Miller theorem. It provides a new way to measure the deviation of real markets from the idealized markets in which the Modigliani-Miller theorem holds.

The structure of the incomplete information risk premium is analogous to the risk premium in reduced form models considered in El Karoui & Martellini (2001) and Jarrow, Lando & Yu (2003). The diffusive premium related to the firm value process corresponds to a premium for diffusive risk in the default intensity process. The event risk premium is analogous to the default event risk premium in intensity based models. However, in the incomplete information setting it is defined in the general reduced form context where an intensity need not exist. Interestingly, Jarrow et al. (2003) show that in the multi-firm intensity based Cox model of Section 3.3, where defaults are conditionally independent, the default event risk premium asymptotically diversifies away.

4.4 Calibration

There is a lively debate in the literature concerning which data should be used to calibrate credit. Jarrow (2001) points to a division between structural and reduced form modelers on this issue. Traditionally, structural models are fit to equity markets and reduced form models are fit to bond markets. Jarrow (2001) argues that the equity and bond data can be used in aggregate to calibrate a credit model and he gives a recipe for doing this in a reduced from setting.

Giesecke & Goldberg (2004*a*) apply the reasoning in Jarrow (2001) to calibrate the I^2 model. The estimation procedure makes use of historical default rates in conjunction with data from equity, bond and credit default swap markets. Huang & Huang (2003) give empirical evidence that structural models yield more plausible results if calibrated to both kinds of data. Importantly, the physical and market-implied probabilities are fit simultaneously. The output of the calibration includes estimates of the risk premium, market implied recovery, model security prices and physical probabilities of default.

One issue addressed in Giesecke & Goldberg (2004*a*) is the relationship between model and actual capital structures. In the classical setting, equity is a European option with strike price and date equal to the face value and maturity of a zero bond. This model is internally consistent. However, it fits market data only to the extent that firm debt can be adequately represented as a zero bond. Giesecke & Goldberg (2004*a*) make use of the flexibility imparted by incomplete information to give a more realistic picture of equity. Specifically, equity is a perpetual down and out call with a stochastic strike price. This approach sidesteps the intractable problem of describing a complex capital structure in terms of a single face value and maturity date.

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