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Calculating Value-at-Risk

by
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Calculating Value-at-Risk [1]

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Abstract: The market risk of a portfolio refers to the possibility of financial loss due to the joint movement of systematic economic variables such as interest and exchange rates. Quantifying market risk is important to regulators in assessing solvency and to risk managers in allocating scarce capital. Moreover, market risk is often the central risk faced by financial institutions.

The standard method for measuring market risk places a conservative, one-sided confidence interval on portfolio losses for short forecast horizons. This bound on losses is often called capital-at-risk or value-at-risk (VAR), for obvious reasons. Calculating the VAR or any similar risk metric requires a probability distribution of changes in portfolio value. In most risk management models, this distribution is derived by placing assumptions on (1) how the portfolio function is approximated, and (2) how the state variables are modeled. Using this framework, we first review four methods for measuring market risk. We then develop and illustrate two new market risk measurement models that use a second-order approximation to the portfolio function and a multivariate GARCH(l,1) model for the state variables. We show that when changes in the state variables are modeled as conditional or unconditional multivariate normal, first-order approximations to the portfolio function yield a univariate normal for the change in portfolio value while second-order approximations yield a quadratic normal.

Using equity return data and a hypothetical portfolio of options, we then evaluate the performance of all six models by examining how accurately each calculates the VAR on an out-of-sample basis. We find that our most general model is superior to all others in predicting the VAR. In additional empirical tests focusing on the error contribution of each of the two model components, we find that the superior performance of our most general model is largely attributable to the use of the second-order approximation, and that the first-order approximations favored by practitioners perform quite poorly. Empirical evidence on the modeling of the state variables is mixed but supports usage of a model which reflects non-linearities in state variable return distributions.

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1 Introduction

Investment and commerical banks, as well as treasury operations of many corporations, hold portfolios of complex securities whose value depends on exogenous state variables such as interest and exchange rat es. To allocate capital, assess solvency, and measure the profitability of different business units (ranging from individual traders to the entire bank), managers and regulators quantify the magnitude and likelihood of possible portfolio value changes for various forecast horizons. This process is often referred to as “measuring market risk”, which is a subset of the risk management function.

It should be noted at the outset that market risk is not the only financial risk faced by banks. For example, banks certainly face financial risk from changes in the credit-worthiness of counter-parties (credit risk), the inability to unload a position in a timely fashion (liquidity risk), errors in implementing trading and pricing policies (execution risk), and other risk factors. Depending on a bank’s strategy and resources, different banks will face each of these risks to varying degrees. It is also possible that these risks are correlated with market risk. However, the purpose of this note is not to identify or quantify all risk factors in detail. Rather, it is to describe the standard methods for measuring market risk and to suggest several ways in which these methods can be improved.

To date, measurement of market risk has focused on one particular metric called value-at-risk, or VAR. The VAR statistic is defined as a one-sided confidence interval on portfolio losses:

$$Prob[\Delta\tilde{P}(\Delta t, \mathbf{\Delta\tilde{x}}) > -VAR] \quad = \quad 1 - \alpha \tag{1}$$

where $\Delta\tilde{P}(\Delta t, \mathbf{\Delta\tilde{x}})$ is the change in the market value of the bank’s *portfolio*, expressed as a function of the *forecast horizon* Δt and the vector of changes in the random *state variables* $\mathbf{\Delta\tilde{x}}$. The parameter α is the *confidence level.* The interpretation is that, over a large number of trading days, the value of the bank’s portfolio will decline by no more than VAR α % of the time. The choice of α depends on the risk tolerance of management and the bank’s excess capital, and is exogenous to our model. Typical values of α range from 1% to 10%. The forecast horizon At is often referred to as an “orderly liquidation period” because it is the time period over which management is confident that underperforming portions of the portfolio can be sold. The choice of At is exogenous to our model. Typical values of Δt range from one day to one week or more.

The preceeding definition of the VAR shows the key role played by the distribution of $\Delta\tilde{P} \equiv \Delta\tilde{P}(\Delta t, \mathbf{\Delta\tilde{x}})$. Calculation of the true distribution of $\Delta\tilde{P}$ is in general not feasible due to the large computational burden involved in revaluing an entire portfolio. Instead, local approximations to $\Delta\tilde{P}$ are made. In this paper we review four such approximations, develop two new ones, then illustrate and test all six by comparing their out-of-sample accuracy in computing the VAR.

The chief contributions of our paper are four-fold. First, we characterize the probability density function of portfolio value changes as a quadratic normal when a second-order model is used to approximate the portfolio function and when a conditional or unconditional multivariate normal distribution is used to model changes in the state variables. Second, we show using an extensive set of options that second-order approximations are always more accurate than first-order approximations. This result is intuitive because second derivatives (gammas) appear in the Black-Scholes partial differential equation. For portfolios of securities without gamma risk, our methods are no more (or less) accurate than delta-based models. Third, we introduce multivariate GARCH modeling into the risk management literature. Previous published papers in risk management have used univariate ARCH models. Such models exclude any notion of correlation or covariance, which is quite limiting for risk management in which the spreading of risk is of central concern. We show empirically that this model is superior on an out-of-sample basis to two others currently in use. Fourth, we empirically demonstrate that our most general model employing a second-order approximation to the portfolio function and a multivariate GARCH(1,1) model is superior to the other five surveyed in computing the VAR for various cr.

Incorporating gamma into a VAR calculation system may also alleviate incentive problems within banks that can arise when the VAR is used to allocate capital. Increasingly, standard practice in investment banks is to evaluate and compensate traders and trading groups on the basis of return per unit VAR. Such an approach may pose incentive problems if the VAR calculation method omits gammas, because traders could, by manipulating gamma, still take significant market risk without changing their VARs. Usage of a risk measurement system that accurately charges for gamma risk may thus discourage attempts to "game" the risk management system when the VAR is used to charge capital. Although banks use additional information to compensate traders and allocate capital, incorporating gamma into a system that provides risk information to both traders and management makes sense, if only because convexity (gamma) considerations appear to play an important role in trading decisions.

This is a timely paper. Recent regulatory and industry advisory committees have strongly recommended that dealers in derivatives adopt formal and regular methods to quantify market risk. In Group of Thirty (1993), for example, an international group of dealers suggested practices and principles for measuring the market risk of portfolios of derivatives. They recommended that dealers calculate a VAR for various confidence levels (e.g., $\alpha = 5\%$ or less) and forecast horizons (of one day or more) as a routine business practice for managing risk. The calculations should be in a framework that reflects risk from all types of securities and all state variables (interest rates, currency exchange rates, commodity prices, and equity prices). These reports, as well as recent comments by regulators, however, do not discuss in detail the appropriate methodology for computing the distribution of portfolio losses; this task is left to the reader. Hence this paper and presumably others.

'In testimony before a U.S. Congressional subcommittee in May of 1994, Alan Greenspan specifically discussed the importance of reflecting non-linearities in risk management: "Although the market risks of many banking instruments can be accurately assessed using simple models, including many derivative contracts, a considerably more sophisticated approach is necessary to assess more complex instruments, especially those with option characteristics, and to aggregate different categories of market risk." This quote is from Chew (1994).

Due to its obvious importance, the technical aspects of bank risk management have begun to attract more attention in the academic literature. For example, Hsieh (1993), Merton and Perold (1994), and Dimson and Marsh (1995), all directly discuss issues related to bank risk management and market risk measurement. Hsieh (1993) is discussed in detail below. Merton and Perold (1994) discuss the computation of risk capital for financial firms. Dimson and Marsh (1995) compare several methods proposed by regulators for computing risk capital for equity portfolios.

The rest of this paper is as follows:

Section 2 briefly reviews four important papers on the measurement of market risk, Garbade (1986), J.P. Morgan (1994), Hsieh (1993), and Wilson (1994). We present simplified versions of these models in a framework that focuses on two key elements: (1) the portfolio function approximation method, and (2) the state variable approximation method. We show that the VAR can be derived by placing assumptions on each of these two elements. Garbade (1986), J.P. Morgan (1994), and Hsieh (1993) assume that changes in the portfolio function can be well-approximated by the delta[3] of the portfolio, but differ in how changes in the state variables are modeled. Garbade (1986) assumes that changes in the state variables over the forecast period can be modeled as a multivariate normal. We will refer to this model as the *delta-normal* model. J.P. Morgan (1994) refines the modeling of the changes in the state variables to reflect non-linearities by weighting the residuals in computing the variance. We will refer to the J. P. Morgan (1994) model as the *delta-weighted normal* model. Hsieh (1993) further generalizes the modeling of the evolution of the state variables with an EGARCH model. We will refer to the Hsieh (1993) model as the *delta-GARCH* model. Wilson (1994) enhances the modeling of the portfolio function but reverts to the Garbade (1986) assumptions on the state variables. Wilson (1994) models the convexity oft he portfolio by explicitly incorporating the gradient and Hessian of the portfolio function. We will refer to this model as the *gamma-normal* model. In Section 6, we compare the out-of-sample performance of these four models using a hypothetical portfolio and equity return data from CRSP.

Section 3 and Section 4 develop our two contributions to market risk measurement. The first model, presented in Section 3, derives a gamma-normal model in a fashion perfectly analogous to how the delta-normal model is derived. For practitioners, this approach may be more intuitive than that of Wilson (1994). Our model may also be more useful because it is computationally simple, relying only on matrix manipulations.

In Section 4, we develop our most general model, which we term *gamma-GARCH*. This model approximates the portfolio function using both the gradient and the Hessian and uses a multivariate

[3]The ***delta*** of a derivative security is the first derivative with respect to the underlying state variable. In a portfolio context in which there are possibly many underlying state variables, the delta refers to the vector of first derivatives of the portfolio with respect to the vector of underlying state variables, also known as the gradient. The gamma of a derivative security is the second derivative with respect to the underlying state variables. In a portfolio contex, the gamma of a portfolio is the matrix of second derivatives of the portfolio with respect to the vector of underlying state variables, also known as the ***Hessian.***

GARCH model to approximate changes in the state variables. This contrasts with Hsieh (1993), which used a univariate ARCH model. We use the Bollerslev (1990) parameterization which assumes a GARCH(1,1) model for the volatility of each state variable and that state variable correlations are constant, conditional on the time-varying volatilities. This model has been successfully applied outside of risk management by Schwert and Sequin (1990), Cechetti, Cumby and Figlewski (1988), Kroner and Sultan (1991), and others. This model is numerically efficient and straightforward to implement.

Earlier authors have not thoroughly investigated the error entailed in using VAR calculation models of the type we discuss. We believe error analysis is especially important because all such models explicitly tradeoff accuracy for computational speed. Our error analysis is conducted in Section 5 in two parts. The first part of Section 5 examines how well each of the portfolio approximation models approximates the portfolio function in the context of computing the VAR. We operate in a univariate setting to keep matters tractable and work with a single European option on a non-dividend paying stock under the assumption that the Black-Scholes option pricing model accurately values the option. As for the state variables, we show in Sections 2 through 4 that the key statistic for the state variables is the covariance matrix of returns next period. Therefore, in the second part of Section 5, we examine the ability of several state variable models to predict, on an out-of-sample basis, the variance of state variable return distributions.

Section 6 compares the accuracy of all six models in calculating the VAR. Using equity data and a hypothetical portfolio, we compare the computed VARs over a thirteen and one-half year test period. We show that our most general model, gamma-GARCH, generally outperforms the others in predicting how frequently actual returns breach the expected VAR on an out-of-sample basis.

Section 7 concludes.

2 Literature Review: Four Risk Measurement Models

We start by presenting four risk measurement models, Garbade (1986), J.P. Morgan (1994), Hsieh (1993), and Wilson (1993), in the context of a two-pronged framework based on (1) how the portfolio function $P(t, \tilde{\mathbf{x}})$ is approximated and (2) how changes in the state variables $(\boldsymbol{\Delta}\tilde{\mathbf{x}})$ are modeled. Our review of these four models is more detailed than usual because we later use these models as benchmarks for evaluating the performance of our own models.

2.1 **The Delta-Normal Model of Garbade** (1986)

We start with the delta-normal risk measurement model of Garbade (1986). In this model, changes in the portfolio function are approximated by the delta of the portfolio (which in this context is the gradient) while changes in the state variables are modeled as a multivariate normal. Using these assumptions and standard multinormal theory, we characterize the approximation to $\Delta\tilde{P}$ as a univariate normal and define and derive a formula for the VAR. In particular, we show that the VAR is a quantile of a univariate normal distribution.

We will use $P(t, \tilde{\mathbf{x}})$ to refer to the portfolio pricing function, with t representing time and $\tilde{\mathbf{x}}$ representing an $n \times 1$ vector of random state variables. For our purposes in risk management, we are only interested in characterizing the distribution of changes in $P(t, \tilde{\mathbf{x}})$ for a short forecast horizon, which we will generically refer to as At. These periods typically range from one day to one week or more. For the state variables, the delta-normal model makes assumptions about returns to holding $\tilde{\mathbf{x}}$ over a forecast horizon Δt.

Assumption X-1. *The sequence of simple returns to holding* $\tilde{\mathbf{x}}$ *over a forecast horizon* Δt, $\mathbf{r}$, *is jointly normally distributed with mean vector* $\mathbf{0}$ *and constant covariance matrix* $\Sigma^{\mathbf{1r}}$, *i.e.,* $\mathbf{r} \sim N_n(\mathbf{0}, \Sigma^{\mathbf{1r}})$, *for* $r_{it} = (x_{i,t+\Delta t} - x_{it})/x_{it}$.

Because the time period is so short, the expected change in the state variables (μ) is assumed to be zero. It will be of use in the sequel to characterize the distribution of $\Delta\tilde{\mathbf{x}}_t \equiv \mathbf{x_{t+\Delta t}} - \mathbf{x_t}$. This is clearly $\Delta\tilde{\mathbf{x}}_t \sim N_n(\mathbf{0}, \mathbf{x}'_{\mathbf{t}}\Sigma^{\mathbf{1r}}\mathbf{x_t})$. We will refer to the quantity $\mathbf{x}'_t\Sigma^{\mathbf{1r}}\mathbf{x}_t$ as $\Sigma^{\mathbf{1}}$. We will refer to **X-1** as the normal or homoskedastic (HOM) model. For $P(t, \tilde{\mathbf{x}})$, the delta-normal model assumes:

Assumption P-1. $P(t, \tilde{\mathbf{x}})$*has one derivative with respect to each argument, denoted* P_t *and* $\mathbf{g}$, *where* $P_t = \frac{\partial P(t,\tilde{\mathbf{x}})}{\partial t}$ *is a scalar and* $\mathbf{g}=[\frac{\partial P(t,\tilde{\mathbf{x}})}{\partial x_1}, \frac{\partial P(t,\tilde{\mathbf{x}})}{\partial x_2}, \cdots, \frac{\partial P(t,\tilde{\mathbf{x}})}{\partial x_n}]$ *is* $n \times 1$. *Higher derivatives are assumed equal to 0.*

We now turn to the derivation of the VAR for the delta-normal model. Recall from (1) that the VAR is defined as a confidence interval on changes in $P(t, \tilde{\mathbf{x}})$ over Δt, i.e., *Prob* $[\Delta\tilde{P}(\Delta t, \Delta\tilde{\mathbf{x}}) > -VAR] = 1 - \alpha$. This definition shows the key role played by $\Delta\tilde{P}$. We start by characterizing the distribution of $\Delta\tilde{P}$ under assumptions **P-1** and **X-1**, then derive the VAR. Under Taylor's Theorem, $P(t, \tilde{\mathbf{x}})$ can be approximated in the neighborhood of $(t_0, \mathbf{x}_0)$ as:

$$
\begin{aligned}
P(t, \tilde{\mathbf{x}}) &= P(t_0, \tilde{\mathbf{x}}_0) + P_t(t - t_0) + \mathbf{g}'[\tilde{\mathbf{x}} - \mathbf{x_0}] + R_2 \\
&= P(t_0, \tilde{\mathbf{x}}_0) + P_t(\Delta t) + \mathbf{g}'\Delta\tilde{\mathbf{x}} + R_2
\end{aligned}
$$

where $P(t_0, \mathbf{x_0})$, is the portfolio's mark-to-market, $\mathbf{g}$ is as defined above, $\Delta\tilde{\mathbf{x}} \equiv (\tilde{\mathbf{x}} - \mathbf{x_0})$ is a vector of (random) state variable changes, $\Delta t \equiv (t - t_0)$ is the forecast horizon, and R_2 is a remainder error term involving second-order and higher derivative terms. Under assumption P-1, $R_2 = 0$. Thus, changes in portfolio value can be approximated by:

$$
\Delta\tilde{P}(\Delta t, \Delta\tilde{\mathbf{x}}) \equiv P(t, \tilde{\mathbf{x}}) - P(t_0, \mathbf{x_0}) \quad = \quad P_t\Delta t + \mathbf{g}'\Delta\tilde{\mathbf{x}}
$$

Hereafter, we will refer to the above first-order approximation of $\Delta\tilde{P}(\Delta t, \mathbf{\Delta\tilde{x}})$ as $\Delta\tilde{P}_1$. We now characterize $\Delta\tilde{P}_1$ as a univariate normal.

Proposition 1 (Characterization of $\Delta\tilde{P}_1$) *$\Delta\tilde{P}_1 \sim N(P_t\Delta t, \mathbf{g}'\mathbf{\Sigma}^1\,\mathbf{g})$.*

Proof: For $\Delta P_1(\Delta t, \mathbf{\Delta\tilde{x}}) = P_t\Delta t + \mathbf{g}'\mathbf{\Delta\tilde{x}}$ we have $E[\,\mathbf{g}'\mathbf{\Delta\tilde{x}} + P_t\Delta t] = P_t\Delta t$ and $Var(P_t\Delta t + \mathbf{g}'\mathbf{\Delta\tilde{x}} = \mathbf{g}'Var(\mathbf{\Delta\tilde{x}})\mathbf{g} = \mathbf{g}'\,\mathbf{\Sigma}^1\,\mathbf{g}$, by **X-1**. The variance does not include terms in Δt because time is non-stochastic. $\Delta\tilde{P}_1$ is normal since the normal family is closed under linear transformations and combinations[4].

We now turn to the calculation of the VAR. Recall that the VAR is defined in $Prob[\Delta P(\Delta t, \mathbf{\Delta\tilde{x}}) > -VAR] = 1 - \alpha$. Using our first-order approximation and the results of the preceeding proposition, we have:

$$\begin{aligned} Prob[\,\frac{\Delta\tilde{P}_1 - P_t\Delta t}{\sqrt{\mathbf{g}'\,\mathbf{\Sigma}^1\,\mathbf{g}}} > \frac{-VAR - P_t\Delta t}{\sqrt{\mathbf{g}'\,\mathbf{\Sigma}^1\,\mathbf{g}}}] &= 1 - \alpha \\ Prob[\,\tilde{Z} > \frac{-VAR - P_t\Delta t}{\sqrt{\mathbf{g}'\mathbf{\Sigma}^1\mathbf{g}}}] &= 1 - \alpha \end{aligned}$$

The first step standardizes $\Delta\tilde{P}_1$ to a unit normal by deducting the mean and dividing by the standard deviation. Solving for the VAR, we obtain:

$$\begin{aligned} Z(\alpha) &= \frac{-VAR - P_t\Delta t}{\sqrt{\mathbf{g}'\,\mathbf{\Sigma}^1\,\mathbf{g}}} \\ VAR &= -P_t\Delta t - Z(\alpha)\sqrt{\mathbf{g}'\,\mathbf{\Sigma}^1\,\mathbf{g}} \end{aligned}$$

where $Z(\alpha)$ is the $100(\alpha)$th percentile of a standard unit normal distribution.

In sum, the delta-normal method combines thetas *(P_t)* and deltas (g), which consist of constants, and the multivariate normality of $\Delta\mathbf{\tilde{x}}$ to characterize the distribution of $\Delta\tilde{P}_1$ as a univariate normal. This yields a highly tractable framework in which it is simple to make probability statements about market risk. Marginals, conditionals, confidence intervals, expected values, and other metrics can then be easily calculated. Although practitioners and regulators have had a tendency to focus exclusively on the VAR, we view the VAR as only a subset of what is possible in such a framework.

[4] more formally, consider the following multinormal theorem from Tong (1990):

Proposition 2 (Normal Linear Transformations and Combinations) *If $\mathbf{w} \sim N_n(\mu, \mathbf{\Sigma})$ and $\mathbf{y} = \mathbf{C}\mathbf{w} + \mathbf{b}$, where $\mathbf{C}$ is any given $m \times n$ real matrix and $\mathbf{b}$ is any $m \times 1$ real vector, then $\mathbf{y} \sim N_m(\mathbf{C}\mu + \mathbf{b}, \mathbf{C}\mathbf{\Sigma}\mathbf{C}')$.*

Replace $\mathbf{w}$ with $\Delta\mathbf{\tilde{x}}$, μ with $\mathbf{0}$, $\mathbf{y}$ with ΔP, $\mathbf{C}$ with $\mathbf{g}$, $\mathbf{b}$ with $\mathbf{0}$, and the argument is closed.

2.2 **The Delta-Weighted Normal Model of J.P. Morgan** (1994)

The delta-weighted normal model, also known as "RiskMetrics," is due to J.P. Morgan (1994)5. As its name implies, the delta-weighted normal model uses the delta to approximate changes in the portfolio function, assumption P-1[6], but differs in how changes in the state variables are modeled, although the key quantity remains Σ, the covariance matrix of changes in the state variables. We will denote the estimate of the covariance matrix under the weighted normal (WTN) model as $\mathbf{\Sigma_t^{2r}}$, which, in general, is time-varying. J. P. Morgan (1994) does not present a formal model of state variable evolution analogous to X-1, but does state clearly and precisely how to compute each element of $\mathbf{\Sigma_t^{2r}}$, $\sigma_{ij(t)}$, given a dataset of returns.

Assumption X-2. *Each element* $\sigma_{ij(t)}$ *of* $\mathbf{\Sigma_t^{2r}}$*, the time-varying covariance matrix of* $\tilde{\mathbf{r}}$*, is computed as* $\sigma_{ij(t)} = \sum_{k=0}^{\infty} \omega_k (r_{i(t-k)} - \mu_{r_i})(r_{j(t-k)} - \mu_{r_j})$ *where* $\omega_k = \lambda^k(1-\lambda)$, $0 < \lambda < 1$,*and* λ *is given exogenously.*

As before it will be of use in computing the VAR to characterize the distribution of $\Delta\tilde{\mathbf{x}}_t \equiv \tilde{\mathbf{x}}_{t+\Delta t} - \tilde{\mathbf{x}}_t$ under **X-2**. This is $\Delta\tilde{\mathbf{x}}_t \sim N_n(\mathbf{0}, \tilde{\mathbf{x}}_t' \mathbf{\Sigma_t^{2r}} \tilde{\mathbf{x}}_t)$. We will refer to the quantity $\mathbf{x}_t' \mathbf{\Sigma_t^{2r}} \mathbf{x}_t$ as $\mathbf{\Sigma_t^2}$.

The weighted-normal (WTN) method differs from the normal (HOM) method in how the squared residuals are used in computing σ_{ij}. Under **X-1**, the residuals receive a weight of $\frac{1}{N}$ in proportion to their contribution to the overall sample (of size *N).* Under X-2, the squared residuals are weighted by exogenous factors ω_k, where k is the time from the observation date to the date the forecast is made. For large samples, these weighting factors sum to one, just as $\frac{1}{N}$ sum to one over any sample of size *N.* The effect of these adjustment factors is to weight recent observations more heavily than observations further away. If t he effect of more recent observations persist, this will be an improvement over the normal method which weights all observations equally. J.P. Morgan (1994) suggests $\lambda = 0.94$ for daily volatility calculations and $\lambda = 0.97$ for monthly volatility estimates. For $\lambda = 0.94$, data more than 75 days away is weighted less than l% as heavily as yesterday's data. In the sequel, we will apply Morgan's method to weekly data. In these calculations we used $\lambda = 0.95$ which we obtained by simple interpolation. Although Morgan does not state the precise reduction in mean-squared prediction error (MSPE) achieved by using exponential weighting over X-1, from charts and tables, it appears as though it may be as much as 50%[7] In practice, an infinite historical time series is not available and the sum is truncated at some reasonable date.

In describing the weighted-normal model, J.P.Morgan (1994) states that λ is calculated to minimize the MSPE but does not provide an explicit algorithm to calculate λ. However, it can be shown

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[5]Their model has many other features (e.g., adjustments for missing data) which we do not discuss. All of the models we present are stripped-down and simplified versions of the actual models designed to convey only the essential points made by the model.

[6]In J.P. Morgan (1994), the VAR calculation omits ***P****ₒ* perhaps because the coefficient of ***P****ₒ* ***At,*** is non-stochastic.

[7]page 63, ***RiskMetrics,*** 1994.

for the weighted-normal method that

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)(x_t - \bar{x}_{t-1})^2$$

or, roughly,

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)u_{t-1}^2$$

Noting that the conventional GARCH(1,1) formulation for σ_t^2 is $\sigma_t^2 = \alpha + \beta\sigma_{t-1}^2 + \gamma u_{t-1}^2$, we find by simple inspection that WTN is approximately GARCH(1,1) with $\alpha = 0$, $\beta = \lambda$, $\gamma = (1-\beta)$, and $0 < \beta < 1$[8]. In fact, the WTN model is seen to be an integrated GARCH, or IGARCH model, without an intercept term.

Because the forecast density remains normal, the calculation of the VAR for the delta-weighted normal method is the same as for the delta-normal method. The only difference is the particular value of Σ used in the forecast. We will illustrate the calculation of the VAR for the delta-weighted normal method in Section 6.

2.3 The Delta-GARCH Model of Hsieh(1993)

As our naming convention implies the delta-GARCH model of Hsieh(1993) uses the delta to approximate changes in the portfolio function and an ARCH model to approximate the changes in the state variables. ARCH models have proven particularly useful in financial econometrics for modeling state variable ret urn distributions, which are known to have persistent variances and heavy tails. ARCH models were first developed by Engle (1982) and were significantly generalized by Bollerslev (1986), Bollerslev (1990), and others. Within the space of numerous ARCH model choices now available, Hsieh (1993) selects the EGARCH parameterization:

$$\begin{aligned} r_t &= \mu + \eta_t \\ \eta_t|\Omega_{t-1} &\sim N(0, h_t) \\ \ln h_t &= \alpha + \beta \ln h_{t-1} + \phi\left[\frac{|\eta_{t-1}|}{\sqrt{h_{t-1}}} - \left(\frac{2}{\pi}\right)^{1/2}\right] + \gamma\frac{\eta_{t-1}}{\sqrt{h_{t-1}}} \end{aligned}$$

where Ω_{t-1} is the information set at $t-1$ and α, β, ϕ, and γ are parameters estimated by maximum likelihood. The use of the natural log insures positivity of the variance without additional restrictions. This is a useful advantage over other ARCH models such as GARCH which require constraints on the parameters that are sometimes difficult to impose in practice. The terms in ϕ and γ comprise a zero mean i.i.d. random sequence and allow the response oft he variance (h_t) to vary depending on whether the shock (η_{t-1}) was negative or positive. This is quite useful for modeling equity return distributions because a large decline in price ("bad news") induces more volatility than a large increase in price ("good news"), a phenomenon originally observed by Black (1976). This feature is of questionable value for modeling exchange rates (Hsieh's focus) because exchange rates are inherently symmetric quantities. If $\phi > 0$ and $\gamma = 0$ then the effect on the variance h_t is then positive (negative) when

[8] page 81, ***RiskMetrics,*** 1995.

$|\eta_{t-1}|$ is larger (smaller) than its expected value, $(2/\pi)^{1/2}$. If $\phi = 0$ and $\gamma < 0$ then the effect on the variance h_t is then positive (negative) when η_{t-1} is negative (positive). EGARCH models were originally proposed by Nelson (1991).

Hsieh's analysis is univariate and consequently excludes any notion of correlation or covariance. This is quite limiting for risk management in which the spreading of risk is of central concern. Therefore, for purposes of implementing and illustrating the delta-GARCH model in this paper, we will use the constant conditional correlation multivariate GARCH model of Bollerslev (1990) instead of Hsieh's univariate EGARCH model. This will make Hsieh's approach meaningful for risk management and will facilitate comparison of the different models. We will discuss the Bollerslev (1990) multivariate parameterization in detail in a later section.

As for the calculation of the VAR for the delta-GARCH model, the only difference from the delta-normal model is the value of $\mathbf{\Sigma}$ used in calculation of the confidence interval. We will denote the estimate of the state variable covariance matrix at time t under the multivariate GARCH model as $\mathbf{\Sigma_t^3}$. For the delta-GARCH model, $\mathbf{\Sigma_t^3}$ varies over time and is forecast using the MSE criterion at time t using all available information. Given $\mathbf{\Sigma_t^3}$, the calculation of the VAR metric is the same as for the delta-normal and delta- weighted normal models.

2.4 The Gamma-Normal Model of Wilson (1994)

In this section we present our interpretation of the gamma-normal model of Wilson (1994). Wilson's model is notable because it is the first published model to reflect the convexity of the portfolio function. In the next section we present our own gamma-normal model. Wilson (1994) utilizes the same assumptions on $\mathbf{\Delta\tilde{x}}$ as the delta-normal model (**X-1**) but uses both the delta and gamma to approximate changes in the portfolio function. We will refer to this assumption as P-2.

Assumption P-2. *$P(t, \tilde{\mathbf{x}})$ has two derivatives wrt t and $\tilde{\mathbf{x}}$, denoted $P_t, P_{tt}, \mathbf{g}, P_{t\tilde{\mathbf{x}}}$, and $\mathbf{H}$. $P_t = \frac{\partial P}{\partial t}$ and $P_{tt} = \frac{\partial^2 P}{\partial t^2}$ are scalars. $\mathbf{g} = \frac{\partial P}{\partial \tilde{\mathbf{x}}}$ and $P_{t\mathbf{x}} = \frac{\partial^2 P}{\partial t \partial \tilde{\mathbf{x}}}$ are $n \times 1$ vectors and $\mathbf{H}$ is an $n \times n$ matrix. Each element of $\mathbf{H}$, $\mathbf{H}_{ij}$, is computed as $\mathbf{H}_{ij} = \frac{\partial^2 P}{\partial x_i \partial x_j}$. Higher order derivatives are equal to 0.*

Wilson (1994, p.74) starts by defining the VAR as "the maximum possible loss over an orderly liquidation period within a given confidence interval." According to Wilson, this definition implies the following optimization program: "solve for the market event that maximizes losses subject to the constraint that the event and all events generating less losses are within a given confidence interval," or:

$$\begin{array}{cc} \max\limits_{\mathbf{\Delta\tilde{x}}} & -\Delta P(t, \mathbf{\Delta\tilde{x}}) \\ \text{s.t.} & F(\mathbf{\Delta\tilde{x}}) \leq \alpha \end{array}$$

where α is the confidence level that the risk manager wants to achieve. Under **X-1** and **P-2**, Wilson asserts this is equivalent to:

$$\begin{array}{ll} \max\limits_{\mathbf{\Delta\tilde{x}}} & -(P_t\Delta t + \mathbf{g}'\mathbf{\Delta\tilde{x}} + \frac{1}{2}\mathbf{\Delta\tilde{x}}'\mathbf{H}\mathbf{\Delta\tilde{x}}) \\ \text{s.t.} & \mathbf{\Delta\tilde{x}}'\Sigma^{-1}\mathbf{\Delta\tilde{x}} \leq \kappa \end{array}$$

P-2 is used to approximate ΔP[9]. The expression $\mathbf{\Delta\tilde{x}}\Sigma^{-1}\mathbf{\Delta\tilde{x}} \leq \kappa$ is equivalent to $F(\mathbf{\Delta\tilde{x}}) \leq \kappa$ under multivariate normality, **X-1**, for suitable α and κ.

Wilson's method is notable because it is the first published model that reflects the convexity of the portfolio function. However, Wilson's method may not be feasible for portfolios with many state variables since the required quadratic program must be solved numerically with an optimization package. Wilson suggests some shortcuts to simplify computation and decrease calculation time but which entail error of unknown magnitude. No benchmarks or examples applying the method are presented.

3 A Competing Gamma-Normal Model

In this section we develop a gamma-normal model that is equivalent to Wilson (1994). Our approach starts with the definition that the VAR is a confidence interval on $\Delta\tilde{P}(\Delta t, \mathbf{\Delta\tilde{x}})$. We then develop an approximation to $\Delta\tilde{P}(\Delta t, \mathbf{\Delta\tilde{x}})$ using **P-2**, which we refer to as $\Delta\tilde{P}_2$. Next, using **X-1**, we characterize the density of $\Delta\tilde{P}_2$ and use established methods to calculate the moments and quantiles of $\Delta\tilde{P}_2$. These steps are perfectly analogous to those used to derive the VAR for the delta-normal model and contrast with the approach of Wilson (1994), which used numerical optimization to compute the VAR or an approximate VAR. We start by recapping our two key assumptions, X-1 and P-2:

Assumption X-1. $\mathbf{\Delta\tilde{x}_t} \sim N_n(0, \mathbf{\Sigma^1})$.

Assumption P-2. *$P(t, \tilde{\mathbf{x}})$ has two derivatives wrt t and $\tilde{\mathbf{x}}$, denoted $P_t, P_{tt}, \mathbf{g}, P_{t\tilde{\mathbf{x}}}$, and $\mathbf{H}$. $P_t = \frac{\partial P}{\partial t}$ and $P_{tt} = \frac{\partial^2 P}{\partial t^2}$ are scalars. $\mathbf{g} = \frac{\partial P}{\partial \tilde{\mathbf{x}}}$ and $P_{t\mathbf{x}} = \frac{\partial^2 P}{\partial t \partial \tilde{\mathbf{x}}}$ are $n \times 1$ vectors and $\mathbf{H}$ is an $n \times n$ matrix. Each element of $\mathbf{H}$, $\mathbf{H}_{ij}$, is computed as $\mathbf{H}_{ij} = \frac{\partial^2 P}{\partial x_i \partial x_j}$. Higher order derivatives are equal to 0.*

We now derive the VAR under these assumptions. We start by expanding $P(t, \tilde{\mathbf{x}})$ in a second-order Taylor series about the mark-to-market of the portfolio. After some matrix manipulation, this yields a quadratic function of a multinormal, which is a density function well-known to statisticians. We then use standard techniques to calculate moments and quantiles, and the VAR follows directly. By Taylor's theorem, $P(t, \tilde{\mathbf{x}})$ can be approximated around the mark-to-market of the portfolio, $P(t_0, \mathbf{x_0})$, as

$$P(t, \tilde{\mathbf{x}}) = P(t_0, \mathbf{x_0}) + P_t\Delta t + \mathbf{g}'\mathbf{\Delta\tilde{x}} + \frac{1}{2}\{\mathbf{\Delta\tilde{x}}'\mathbf{H}\mathbf{\Delta\tilde{x}} + 2P_{t\mathbf{x}}\mathbf{\Delta\tilde{x}}\Delta t + P_{tt}(\Delta t)^2\} + R_3$$

[9]Wilson omits $P_{t\tilde{\mathbf{x}}}$ and P_{tt} from his analysis, perhaps due to immateriality or the fact that neither term appears in the Black-Scholes PDE.

Now under P-2, R_3 equals O, so we have:

$$\begin{aligned}\Delta\tilde{P}_2(\Delta t, \mathbf{\Delta\tilde{x}}) &\equiv P(t, \tilde{\mathbf{x}}) - P(t_0, \tilde{\mathbf{x}}_0) \\ &= P_t\Delta t + \mathbf{g}'\mathbf{\Delta\tilde{x}} + \frac{1}{2}\{\mathbf{\Delta\tilde{x}}'\mathbf{H}\mathbf{\Delta\tilde{x}} + 2P_{t\mathbf{x}}\mathbf{\Delta\tilde{x}}\Delta t + P_{tt}(\Delta t)^2\}\end{aligned}$$

where we have used $\Delta\tilde{P}_2$ to refer to the approximate change in the portfolio function under assumption **P-2**. $\Delta\tilde{P}_2$ is simply a quadratic function of a multinormal vector, $\mathbf{\Delta\tilde{x}}$. This is a density function well-known to statisticians. We next discuss how to calculate moments of a general quadratic form of a multinormal, which we will use to calculate the quantiles of $\Delta\tilde{P}_2$ and therefore the VAR. For $\tilde{\mathbf{y}} \sim N_p(\mu, \Sigma)$, with $Q(\tilde{\mathbf{y}}) = \tilde{\mathbf{y}}'\mathbf{A}\tilde{\mathbf{y}} + \mathbf{a}'\tilde{\mathbf{y}} + d$, $\mathbf{A} = \mathbf{A}'$, the r-th moment of $Q(\tilde{\mathbf{y}})$ is given by:

$$\begin{aligned}E(Q(\tilde{\mathbf{y}}))^r &= \left[\sum_{r_1=0}^{r-1}\binom{r-1}{r_1}g^{r-1-r_1}\sum_{r_2=0}^{r_1-1}\binom{r_1-1}{r_2}g^{r_1-1-r_2}\cdots\right] \\ \text{where}\quad g^{(k)} &= \frac{1}{2}k!\sum_{j=1}^{p}(2\lambda_j)^{k+1} + \frac{(k+1)!}{2}\sum_{j=1}^{p}b_j^2(2\lambda_j)^{k-1},\quad k \geq 1 \\ &= \frac{1}{2}\sum_{j=1}^{p}(2\lambda_j) + (d + \mathbf{a}'\mu + \mu'\mathbf{A}\mu), \qquad k = 0\end{aligned}$$

where $\mathbf{P}'\mathbf{\Sigma}^{1/2}\mathbf{A}\mathbf{\Sigma}^{1/2}\mathbf{P} = diag(\lambda_1, \ldots, \lambda_p) = \mathbf{\Lambda}, \mathbf{PP}' = \mathbf{I}, \mathbf{P}'(\mathbf{\Sigma}^{1/2}\mathbf{a} + 2\mathbf{\Sigma}^{1/2}\mathbf{A}\mu) = \mathbf{b} = (b_1, \ldots, b_p)'$. A more comprehensible version of the above formula for the first few moments is as follows:

$$\begin{aligned}E[Q(\tilde{\mathbf{y}})] &= \mu_1 = g^{(0)} \\ E[Q(\tilde{\mathbf{y}})]^2 &= \mu_2 = \binom{1}{0}g^{(1)}\mu_0 + \binom{1}{1}g^{(0)}\mu_1 \\ E[Q(\tilde{\mathbf{y}})]^3 &= \mu_3 = \binom{2}{0}g^{(2)}\mu_0 + \binom{2}{1}g^{(1)}\mu_1 + \binom{2}{2}g^{(0)}\mu_2 \\ E[Q(\tilde{\mathbf{y}})]^4 &= \mu_4 = \binom{3}{0}g^{(3)}\mu_0 + \binom{3}{1}g^{(2)}\mu_1 + \binom{3}{2}g^{(1)}\mu_2 + \binom{3}{3}g^{(0)}\mu_3\end{aligned}$$

where μ_0 is taken to be 1. These results are from Mathai and Provost (1992). To compute the moments for our quadratic form $\Delta\tilde{P}_2$ we use $\tilde{\mathbf{y}} \sim N(\mathbf{0}, \Sigma)$, $\mathbf{A} = \frac{1}{2}\mathbf{H}$, $\mathbf{a} = (\mathbf{g} + P_{tx}\Delta t)$, and $d = P_t\Delta t + P_{tt}(\Delta t)^2$. Preliminary numerical results indicated that omission of the terms in P_{tx} and P_{tt} did not significantly affect the results. Therefore, in all of the examples that follow, we omitted these terms from the computations.

We now turn to a discussion of how to tabulate the distribution function of our quadratic form, $F_{\Delta\tilde{P}_2}$. No closed form expression $F_{\Delta\tilde{P}_2}$ exists and, because of the large number of parameters involved, few tables have been compiled. Therefore, we developed our own software to tabulate $F_{\Delta\tilde{P}_2}$ using several published met hods, most of which were developed by Ruben (1962,1963). All of these techniques are approximate and are based on expansions of $F_{\Delta\tilde{P}_2}$ into infinite sums of the distribution functions of simpler but related univariate distributions (e.g., the chi-square or non-central chi-square) or of special functions (e.g., Laguerre or Hermite polynomials). We implemented and tested several of these methods, and others, and used sample tables that were published in various papers to validate our software. However, we found that these methods were either too slow or experienced machine underflow when tested wit h plausible parameters. Ultimately, we settled on Cornish-Fisher (CF) expansions to calculate quantiles, as described by, for example, Johnson and Kotz (1970) or Kendall, Stuart, and Ord (1987). CF expansions express the quantile of a distribution (e.g., $F_{\Delta\tilde{P}_2}(\alpha)$) in terms of its cumulants and the corresponding quantile of the standard normal distribution, $\Phi(\alpha)$:

$$F_{\Delta\tilde{P}_2}(\alpha) = \Phi(\alpha) + \frac{1}{6}(\Phi(\alpha)^2 - 1)k_3 + +\frac{1}{24}(\Phi(\alpha)^3 - 3\Phi(\alpha))k_4 - \frac{1}{36}(2\Phi(\alpha)^3 - 5\Phi(\alpha))k_3^2$$

where k_3 and k_4 are the third and fourth cumulants of $\Delta\tilde{P}_2$[10]. The preceeding expression assumes that $\Delta\tilde{P}_2$ has been standardized. Standard methods are used to compute $\Phi(\alpha)$ (e.g., Abramowitz and Stegun (1972)). We found this method to be both reliable and accurate in extensive simulation testing.

3.1 Relation to Wilson (1994)

We remarked earlier that our method and the method of Wilson (1994), though clearly different in design and derivation, are equivalent. Here we demonstrate this in a simplified univariate setting. To facilitate an analytical solution in both frameworks, we construct a stylized example and focus on calculating one metric, the VAR at 5%. We assume one state variable, $\tilde{s}$, with $\tilde{s} \sim N(0,1)$, $P_{ss} = \Gamma = 1$, $P_s = \delta = 4$ and $P_t = \theta = 4$. Other derivatives are assumed equal to 0. This implies $\tilde{Q} \equiv \Delta P(\tilde{s}) = \tilde{s}^2 + 4\tilde{s} + 4$. Our approach explicitly characterizes the density function of $\tilde{Q}$. Our first step is to complete the square:

$$\tilde{Q} = \tilde{s}^2 + 4\tilde{s} + 4 = (\tilde{s} + 2)^2.$$

We recognize this expression as a characterization of a non-central χ^2 distribution with $\nu = 1$ degrees of freedom and non-centrality parameter $\lambda = 2^2 = 4$. This distribution is a special case of the quadratic normal. Using a table of critical values, we find the 5% quantile for this distribution as –13.28.

Wilson's approach computes a VAR at 5% by maximizing the negative of the change in portfolio value ($-\Delta P(s)$), subject to the constraint that the standard normal random variable $\tilde{s}$ move only within it's 5% confidence bands:

$$\max_{s} \quad -Q = -s^2 - 4s - 4$$
$$\text{s.t.} \quad 0.05 < F_Z(s) < 0.95$$

[10] For any distribution X, the sth cumulant of X, k_s, is defined in $ln\,(M_X(t)) = \sum_{s=1}^{\infty} k_s \frac{t^s}{s!}$ where $M_X(t)$ is the moment generating function of of X, i.e., $M_X(t) = E[e^{tX}]$. Just as the moments (μ_s) are the coefficients of $\frac{t^s}{s!}$ in an expansion of $M_X(t)$, the cumulants (κ_r) are the coefficients of $\frac{t^r}{r!}$ in an expansion of $ln(M_X(t))$. For further information, see Johson & Kotz (1970) or Kendall, Stuart, and Ord (1987).

We can solve this optimization problem by inspecting the value of the function at either endpoint of the constraint and at the minimum value of the function in between. The minimum value of the function occurs at Z(s) = 0.95 or s = 1.645, where –Q = – 13.28. This figure matches the value we calculated. Therefore, the two methods are equivalent, at least in this simplified univariate setting.

In summary, we have developed a gamma-normal model that has numerous advantages over Wilson (1994). Our method is more intuitive because our approach explicitly characterizes the distribution of $\Delta\tilde{P}_2$. As such, this approach is a direct generalization of the delta-normal and will likely be more accessible to practitioners. Computation of the VAR is also simpler with our method. With Wilson's method, one must numerically solve an optimization problem, or make some simplifying assumptions (e.g., diagonal H) that entail unknown error. The high dimensionality of the optimization problem may pose computational difficulties in practice. In contrast, we provide quick formulae to calculate the VAR without simplifying assumptions.

4 Gamma-GARCH Model

In this section we derive our most general model, which we term *gamma-GARCH*. This model uses a second-order approximation to the portfolio function and a multivariate GARCH model to approximate changes in the state variables. We start by stating our assumptions, P-2 and X-3. P-2 is identical to the portfolio function assumption used for the gamma-normal model of Wilson (1994) and our own in the preceding section

Assumption P-2. *$P(t, \tilde{\mathbf{x}})$ has two derivatives wrt t and $\tilde{\mathbf{x}}$, denoted $P_t, P_{tt}, \mathbf{g}, P_{t\tilde{\mathbf{X}}}$, and $\mathbf{H}$. $P_t = \frac{\partial P}{\partial t}$ and $P_{tt} = \frac{\partial^2 P}{\partial t^2}$ are scalars. $\mathbf{g} = \frac{\partial P}{\partial \tilde{\mathbf{X}}}$ and $P_{t\mathbf{X}} = \frac{\partial^2 P}{\partial t \partial \tilde{\mathbf{X}}}$ are $n \times 1$ vectors and $\mathbf{H}$ is an $n \times n$ matrix. Each element of $\mathbf{H}$, $\mathbf{H}_{ij}$, is computed as $\mathbf{H}_{ij} = \frac{\partial^2 P}{\partial x_i \partial x_j}$. Higher order derivatives are equal to 0.*

For the state variables, we utilize the multivariate GARCH parameterization suggested by Bollerslev (1990), which we label X-3

Assumption X-3. *State variable returns, $\tilde{\mathbf{r}}_\mathbf{t}$ are distributed as:*

$$\begin{aligned} \tilde{\mathbf{r}}_\mathbf{t} &= \mu + \mathbf{u}_t \\ \mathbf{u}_t | \Psi_{t-1} &\sim N(0, \mathbf{\Sigma}_\mathbf{t}^{\mathbf{3r}}) \end{aligned}$$

where $\tilde{\mathbf{r}}, \mu, \mathbf{u}_t$ are $n \times 1$ and $\mathbf{\Sigma}_\mathbf{t}^{\mathbf{3r}}$ is $n \times n$. Ψ_{t-1} is the information set at $t-1$. Individual elements of $\mathbf{\Sigma}_\mathbf{t}^{\mathbf{3r}}$ are given as follows:

$$\begin{aligned} v_{ii(t)} &= \alpha_{0i} + \alpha_{1i} u_{i,t-1}^2 + \beta_i v_{ii(t-1)} \\ \textit{and} \qquad v_{ij(t)} &= \rho_{ij} \sqrt{v_{ii(t)}} \sqrt{v_{jj(t)}}. \end{aligned}$$

The covariance matrix can be written as $\mathbf{\Sigma}_\mathbf{t}^{\mathbf{3r}} = \mathbf{V_t}\,\mathbf{R}\,\mathbf{V_t}$ where $\mathbf{V_t}$ is a time-varying diagonal matrix with typical element $\sqrt{v_{ii(t)}}$, and $\mathbf{R}$ is a time-invariant matrix with typical element ρ_{ij}. The principal advantages of this multivariate parameterization over others that have been suggested are parsimony and simplicity. Both features stern from the assumption of constant correlation, conditional on the univariate GARCH(1,1) models for the volatilities. As before, it will be useful to characterize the distribution of $\Delta\tilde{\mathbf{x}}_t$ under **X-3** as $\Delta\tilde{\mathbf{x}}_t \sim N_n(\mathbf{0}, \mathbf{\Sigma}_t^3)$, with $\mathbf{\Sigma}_t^3 = \mathbf{x}_t'\mathbf{\Sigma}_t^{3r}\mathbf{x}_t$.

Estimation is via numerical maximum likelihood using the Berndt, Hall, Hall, and Hausmann (1974) algorithm with starting parameters determined by a quasi-random grid search. Standard errors are acheived as the square root of the diagonal of the inverse of the Hessian matrix evaluated at the MLE.

Calculation of the VAR is basically the same as before for the gamma-normal model. The forecast density for the state variables is still multinormal. The only difference is that the estimate of $\mathbf{\Sigma}$ is calculated according to the GARCH model conditional on the available information at time *t*. Because there is no change in the modeling of the approximation to the portfolio function, the forecast density for $\Delta\tilde{P}_2$ is still quadratic normal, and the methods and formulae developed earlier for computing the VAR and other statistics are the same as for the gamma-normal method.

Note that the gamma-GARCH model does not include the others as special cases. In particular, the WTN model features time-varying conditional correlations, which are not part of the GARCH model. Of course, the delta approximation is clearly a special case of the gamma approximation when $\mathbf{H}=\mathbf{0}$, $P_{t\mathbf{x}}=\mathbf{0}$, and $P_{tt} = 0$.

5 Error Analysis

The models we have described and developed in the preceding sections are all explicitly local approximations. In this section we conduct tests to examine how accurately both parts of the model perform. The first subsection examines the accuracy of the two portfolio approximation methods in computing the VAR. We operate in a univariate setting to keep matters tractable and work with a single European option on a non-dividend paying stock under the assumption that the Black-Scholes option pricing model accurately values the option. We show that the gamma method is far more accurate than the delta method according to a root mean square percentage error (RMSPE) criterion. Of the 120 options we use as test cases, the gamma method outperforms the delta method in all instances. Moreover, the error of the gamma approximation is fairly constant for decreasing α. Both models perform poorly for short-dated, deep out-of-the-money options, which are essentially valueless.

We showed in Sections 2 through 4 that the key quantity for the state variables is the covariance matrix of returns next period. Therefore, the second subsection examines the out-of-sample performance of each of the three models (HOM, WTN, and GARCH) in forecasting the covariance matrix of state variable returns. We use equity return data because of the availability y of CRSP, a high-quality database with weekly data spanning more than 25 years. To evaluate the out-of-sample relative performance of the three models, we use two methods. The first examines the square root of the mean squared prediction error (RMSPE) of the square of the return of the state variables; this provides mixed results. The second is a standard regression test of efficiency that regresses the squared realized return on the predicted squared return. In such a regression, an unbiased and efficient forecast should yield an intercept of zero and a slope coefficient of one with serially uncorrelated errors and a high R^2. On this basis, the GARCH model marginally outperforms the WTN model. The HOM model is vastly inferior to both WTN and GARCH by this metric. This combination of tests has been used by Cho and West (1994), Canina and Figlewski (1993), Jorion (1995), and others.

5.1 Error Analysis for Approximation of the Portfolio Function

This section evaluates the performance of first-order (delta) and second-order (gamma) models in computing the VAR. We operate in a univariate setting to keep matters tractable and work with a single European option on a non-dividend paying stock under the assumption that the Black-Scholes option pricing model accurately values the option. For each experiment we perform, we assume a risk free interest rate of 7%, an underlying stock volatility of 20%, and a VAR forecast period of one week ($\Delta t = 1/52$).

Tables 1 through 6 show six analyses of the approximation error. The first three show the error and percentage error of the delta and gamma models when the portfolio consists of a single European put at three different α levels, $\alpha = 10\%$, 5%, and 1%. The following three tables show the same results for a European call. The initial option position in each case is normalized to \$1,000 to facilitate comparison. For informational purposes, each table shows the values of the key inputs $(P_t, P_x$ and $P_{xx})$ and the option premium (P). All six tables show results for four different times to maturity (τ = 1.00, 0.50, 0.25, 0.10) and five different values of option 'moneyness'[11] (1.00/1.10, 1.00/1.05, 1.00, 1.05, and 1.10).

We computed all values using simulation and the true distribution of security returns under risk-neutrality. Usage of the true distribution isolates the error introduced by the portfolio approximation methods from any possible error caused by improper modeling of the state variables. The appropriate risk-neutral density for the stock price next period is:

$$\tilde{S}_{t+\Delta t} = S_t \, e^{\{(r-\sigma^2/2)\Delta t + \tilde{z}\sigma\sqrt{\Delta t}\}}$$

where $\tilde{z}$ is a standard unit normal. For each simulated value of $\tilde{S}_{t+\Delta t}$, say $S^i_{t+\Delta t}$, we computed the following quantities:

$$\begin{aligned}
\Delta P_T(S^i_{t+\Delta t}) &= BS(S^i_{t+\Delta t}, t+\Delta t) - BS(S_t, t) \\
\Delta P_1(S^i_{t+\Delta t}) &= \frac{\partial BS(S_t,t)}{\partial t}\Delta t + \frac{\partial BS(S_t,t)}{\partial S}(S^i_{t+\Delta t} - S_t) \\
\Delta P_2(S^i_{t+\Delta t}) &= \frac{\partial BS(S_t,t)}{\partial S}(S^i_{t+\Delta t} - S_t) + \frac{\partial BS(S_t,t)}{\partial t}\Delta t + \\
&\quad \frac{1}{2}\frac{\partial^2 BS(S_t,t)}{\partial t \partial t}(\Delta t)^2 + \frac{\partial^2 BS(S_t,t)}{\partial S \partial t}(\Delta t)(S^i_{t+\Delta t} - S_t) + \frac{1}{2}\frac{\partial^2 BS(S_t,t)}{\partial S \partial S}(S^i_{t+\Delta t} - S_t)^2
\end{aligned}$$

where $BS(S,t)$ is the Black-Scholes option pricing formula. Given a simulated draw of $S^i_{t+\Delta t}$, we used ΔP_T, ΔP_1 and ΔP_2 as individual observations from the distribution of the true change in portfolio value, the distribution under a first-order approximation, and the distribution under a second-order approximation, respectively. Using 10,000 draws for $S^i_{t+\Delta t}$, we then calculated the VARs for each of the three methods by selecting the α% quantile from each of the three empirical distributions. Standard errors were less than 1% in most instances.

We provide three metrics to compare the performance of the two approximation methods: (1) mean percent age error (MPE), defined as the average percentage difference between the true VAR and an approximate e VAR; (2) mean absolute percentage error (MAP E), defined as the average difference between the absolute values of the percentage difference between the true VAR and an approximate VAR; and (3) the square root of the mean squared percentage error (RMSPE), defined as the square root of the average of the square of the percentage difference between the true VAR and an approximate VAR. We will refer to the average error under these three metrics (at a confidence level α) as $MPE(\alpha)$, $MAPE(\alpha)$, and $RMSPE(\alpha)$, respectively.

It turns out the choice of metric makes no difference, because, according to all three, the gamma method always outperforms the delta method, usually by wide margins. The percentage error of the

[11] We define the *moneyness* of an option as the ratio of the underlying stock price (S) to the option strike price (K), ($\frac{S}{K}$).

gamma method is low, usually less than 10%, except in pathological situations where the option is essentially without value.

Table 1

Table 2

Table *3*

In Table 1, for a European put at $\alpha = 10\%$, the MAPE for the gamma method (the "gamma MAPE") is one-third of the delta MAPE (5% vs. 15%). Moreover, the relative performance of the gamma method over the delta method improves the smaller the α, i.e., the further away from the mark-to-market of the portfolio. For example, while the delta MAPE clearly increases for decreasing α (MAPE(10%)=21%, MAPE(5%)=29%, MAPE(1%)=45%), the gamma MAPE remains roughly constant (MAPE(10%)=5%, MAPE(5%)=5%, MPE(1%)=6%). The results are similar on a RMSPE basis. At $\alpha = 10\%$, the delta and gamma RMSPE are 31% and 11%, respectively, while at $\alpha = 1\%$, the delta and gamma RMSPE are 63% and 10%, respectively.

For calls, the performance of the delta method improves, but on average is still much inferior to the performance of the gamma method. As before, while the error of the delta method clearly increases as α is decreased (MAPE(10%)=15%, MAPE(5%)=21%, and MAPE(1%)=32%), the MAPE of the gamma method remains small and constant (MAPE(10%)=6%, MAPE(5%)=6%, and MAPE(1%)=6%).

Table 4

Table *5*

Table *6*

The MPE, MAPE, and RMSPE reported in Tables 1 through 6 are summarized in Table 7.

Table 7

For all 120 options that we examine, the error of the delta method is negative. Thus the delta method underestimates the VAR and is too conservative. The effect is stronger the smaller the α. This underestimation of the VAR stems from the fact that gamma is positive for all of the option cases we happen to examine. For other securities with negative convexity (gamma), the delta method would overestimate the VAR, *ceteris paribus.*

Both models perform poorly when the option is essentially valueless. These cases occur when the option is short-dated ($\tau = 0.10$) and deep out-of-money ($\frac{S}{K}$=1.10 for puts and $\frac{S}{K}$=1/1.10 for calls). In these instances, the gamma MAPE averages less than .50%, while the corresponding delta MAPE averages more than 100%.

Our conclusion is that the delta method performs very poorly and that the gamma method is far superior to the delta method. This result is particularly intuitive in the context of derivative asset pricing since second derivatives appear in the the Black-Scholes PDE. The performance of the delta method is slightly better for calls than for puts, but still much inferior to the performance of the gamma met hod. The error for the gamma method is slightly positive for puts, slightly negative for calls. The delta met hod is always too conservative (estimates a VAR more negative than the real VAR) because gamma is always positive in the cases we examine. Both models perform poorly for options which are essentially valueless.

5.2 Forecast and Error Analysis of Σ

5.2.1 State Variable Data

Table 8 shows an excerpt of the data we used in our analysis. To simplify matters, we used only equity data from CRSP, a database that is well-known and of high-quality.

Table *8*

We used weekly data sampled on successive Wednesdays to help mitigate problems others have encountered with daily data. These include spurious negative serial correlation due to "bid-ask" bounce, as well as problems due to non-trading periods such as holidays, which normally occur on Mondays, Tuesdays, Thursdays, and Fridays. We found usage of weekly data (with weeks beginning and ending on successive Wednesdays) largely avoids these problems. The equity prices are contemporaneous daily closing prices for Dow Chemical (DOW), Exxon (EXX), Union Carbide (UNCA), Coca-Cola (COKE), and the Standard and Poors 500 Index (S&P 500) over a twenty-six year period starting on January 1, 1969 and ending on December 31, 1994. These five securities are widely held, actively traded, and highly liquid. To compute a weekly return series using the twenty-six years of daily data, we compounded daily returns between successive Wednesdays. This approach netted 1356 observations for each series.

Of course, these price series are for illustrative purposes only. Other equity price series and other state variable types (e.g., currency prices, interest rates, commodity prices) could be incorporated into the framework without difficulty or loss of generality. We selected a forecast time period of one week (Δt=1/52) to illustrate and test the three models. Other time horizons (both shorter and longer) could be incorporated into the framework without difficulty or loss of generality.

In computing $\mathbf{\Sigma^{1r}}$, $\mathbf{\Sigma_t^{2r}}$, and $\mathbf{\Sigma_t^{3r}}$, we used data in a similar way. We first divided the database into two periods, 1969-1980.5, and 1980.5-1994. We used the first period of twelve and one-half years (650 weekly observations) to calibrate our models. We found this time period was necessary to allow the parameters of the GARCH model to stabilize. Each time we recomputed parameters thereafter, we used the most recent 650 weekly observations then available. The last thirteen and one-half years of data (706 weekly observations) were used for out-of-sample testing. In testing the models over this period, we calculated covariance matrices for the HOM, WTN, and GARCH models on a weekly basis. For the GARCH models, we recalculated parameters only twice a year because the numerical maximization was quite time consuming. Of course, in forecasting the value of $\mathbf{\Sigma_t^{3r}}$ at time $t-1$, we used the most recent conditional variances and squared realized returns, as called for under X-3.

5.2.2 Calculation of Σ^{1r} Under X-1

$\mathbf{\Sigma^{1r}}$ is used as an input to the delta-normal and gamma-normal models. Under **X-1**, our assumptions on $\mathbf{\Delta\tilde{x}}$ satisfy the OLS conditions, so each element of $\mathbf{\Sigma^{1r}}$, σ_{ij}, can be calculated via simple linear regression of $\tilde{r}_i$ on $\tilde{r}_j$. In particular, the following regression model is suggested:

$$\begin{aligned} \tilde{\mathbf{r}}_{\mathbf{t}} &= \mu + \tilde{\epsilon}, \qquad \tilde{\epsilon} \sim N_n(\mathbf{0}, \mathbf{\Sigma^{1r}}) \\ \text{for } \tilde{r}_{it} &= \frac{x_{it} - x_{it-\Delta t}}{x_{it-\Delta t}}, \qquad i \,\epsilon\, \{DOW, EXX, UNCA, COKE, S\&P500\} \end{aligned}$$

where Δt is the forecast horizon and x_{it} is a price series. As noted earlier, we used the return series available directly from CRSP. The differencing interval is identical to the forecast horizon, 1 week. Table 9 shows the OLS covariance matrix, $\mathbf{\Sigma^{1r}}$, as of 12/28/94 for the five state variables. All values were highly significant.

Table 9

5.2.3 Calculation of Σ_t^{2r} Under X-2

Table 10 shows the results of applying J.P.Morgan's model to compute $\mathbf{\Sigma_t^{2r}}$ as of 12/28/1994.

Table 10

5.2.4 Calculation of Σ_t^{3r} Under X-3

Table 11 shows parameter estimates under X – 3 for the regression model X – 3 as of 12/28/1994.

Table 11

5.2.5 Evaluation of the Three Forecasting Methods

Following Jorion (1995), Cho and West (1994), Canina and Figlewski (1993), and numerous other authors, we use two metrics to evaluate the performance oft he three state variable models in forecasting the covariance matrix of returns next period: (1) the square root of the mean squared prediction error (RMSPE); and (2) the results of regression efficiency tests of the realized squared return on the forecast variance. The RMSPE is computed as:

$$RMSPE = \sqrt{\frac{1}{T}\sum_{1}^{T}(r_{it}^2 - \hat{\sigma}_{it}^2)^2}$$

where T is the number of forecasts available, r_{it}^2 is the square of the return realized at date t, and $\hat{\sigma}_{it}^2$ is the forecast of the variance at time t, computed at $t-1$. Using the forecast variances computed under X-1, X-2, and X-3, we applied the RMSPE formula for each of the three models at each of the 706 forecast dates. Results are shown in Table 12.

Table 12

The evidence in Table 12 is inconclusive. According to the RMSPE metric, the GARCH model appears only marginally superior to WTN and HOM. For the S&P 500, the RMSPE for HOM, WTN, and GARCH, are 0.00158, 0.00155, and 0.00154, respectively. These results are representative of the other state variables. Thus, no model emerges as clearly superior.

An alternative technique used by researchers is to conduct an efficiency regression of the square of realized returns on the forecast variance,

$$r_t^2 = b_0 + b_1\hat{\sigma}_t^2 + \epsilon_t.$$

The forecast variance for time t, $\hat{\sigma}_t^2$, would be computed according to each of the three state variable models at $t-1$. If $\hat{\sigma}_t^2$ is an *unbiased* forecast, then $E[\hat{\sigma}_t^2] = E[r_t^2]$. This implies the hypothesis test $H_0 : b_0 = 0, b_1 = 1$. If $\hat{\sigma}_t^2$ is an *efficient* forecast, then the sequence ϵ_t should be white noise and serially uncorrelated. This implies a Durbin-Watson test for serial correlation. Moreover, the higher the $R*$ in such a regression, the more information contained in the forecast. Results of these regressions appear in Table 13.

Table 13

The GARCH regression coefficients appear plausible. In the case of the S&P 500, the evidence for $H_0 : b_0 = 0, b_1 = 1$ is strongest for the GARCH model among all 15 regressions. Most other b_1 for GARCH hover around 1.00. For HOM, the b_1 coefficient values appear random. The WTN b_1 coefficients are similar among the different regressions, but are all too low and average 0.68. Recall that J.P. Morgan (1995)'s WTN model is equivalent to a zero intercept IGARCH(l,l) model with regression parameter λ set equal to 0.95, exogenously, for all models. The consistent but low results for the WTN b_1 coefficients may indicate that the underlying IGARCH model is correctly specified but that the choice of $\lambda = 0.95$ is not optimal.

The Durbin-Watson test indicates that the strongest evidence against serial correlation (and therefore in favor of efficiency) is for the S&P 500 GARCH regression. In this case, the Durbin-Watson statistic is 1.94 versus an ideal value of 2.00. However, two of the five GARCH models fail the Durbin-Watston test while both WTN and HOM generally pass (at 1% and 5% levels) when the GARCH model passes.

By the R^2 standard, both the WTN and GARCH models outperform HOM by wide margins, with the GARCH model marginally superior to WTN. For the S&P 500, the R^2 for the GARCH, WTN, and HOM models are 0.048, 0.036, and 0.006, respectively. These results are representative of the other four sets of regressions.

Thus, while the evidence slightly favors the GARCH model over the two alternatives, none emerges as a clear best choice. This contrasts with our anlaysis of the portfolio function approximation methods in which the gamma method emerged as unambiguously superior to the delta method. The poor performance of the HOM model is to be expected inasmuch as many studies have shown that volatility is time-varying. The performance of the WTN model might be improved upon by optimizing on the choice of λ. Moreover, the performance of all three models likely suffers from the fact that none permit the variance to respond asymmetrically to positive and negative r_{it}.

[Ongoing research tests other state variable models, including the EGARCH model of Nelson (1991), an integrated GARCH model (IGARCH), an IGARCH model without an intercept term, and the GARCH model of Glosten, Jaganathan, and Runkle (1992) (GJRGARCH). The IGARCH model without an intercept is the J.P. Morgan model without the constraint $\lambda = 0.95$. EGARCH and GJRGARCH permit asymmetric responses of the volatility to the sign of r_t.]

6 Illustration and Comparison of the Methods

In this section we illustrate and compare the performance of the six risk measurement models. These six models arise from considering all combinations of the two portfolio approximation models (P-1 and P-2) and the three state variable models (X-1, X-2 and X-3). The data source and state variables are the same as earlier. Our period of analysis is 01/01/1969 through 12/31/1994, a period of twenty-six years. As above, we used the first twelve and one-half years (650 weeks) to calibrate the models and the remaining thirteen and one-half years (706 weeks) for out-of-sample testing.

Table 14 lists the five securities we used to test our model. These are standard European options, three puts and two calls, one on each state variable. We used the Black and Scholes (1973) option pricing model to value these securities and to calculate hedge ratios. We set the input parameters for all five options to achieve a range of values for the delta vector (g) and gamma matrix **(H)** to illustrate the flexibility of our method. We tested a variety of portfolios and found that the results were always very similar to those we report. At each forecast date, we set the value per state variable to $200 by adjusting the number of options on each state variable. This made the total value of the portfolio $1,000 and facilitated comparison among different days. We used these securities and this model because of the known convexity properties of options and the universality of the Black-Scholes model. In applying this model to each of the state variables, we did not make modifications to the formulae for dividends received on the equities and other possible real-world complications such as transactions costs. We view these refinements as unnecessary for an analysis of this type and do not believe they would affect the basic conclusions of our comparison of the alternative market risk measurement methods. In addition, the option pricing parameters assumed are clearly hypothetical.

Table 14

6.1 Sample VAR Results for One Week

The results for one week are presented in Table 15. We show seven statistics for each of the six VAR measurement methods, the first four central moments (the mean, the variance, the skewness, and the excess kurtosis) and the first, fifth, and tenth percentiles. Of course, the latter three of these statistics are the VAR at l%, 5%, and 10% confidence levels. Skewness and excess kurtosis are zero for all of the delta models because in all cases the forecast density is univariate normal. The corresponding quadratic normal densities are both positively skewed and leptokurtotic.

Table 15

6.2 Out-of-Sample VAR Comparison

Table 16 compares the performance of the six VAR calculation methods in correctly predicting the VAR. Three α values are tested, 10%, 5%, and 1%. For each expected α level for each model, we show the actual number of times over the 706 week testing period that the actual realized loss on the portfolio exceeded the estimated VAR. This count is denoted as C in Table 16. Next to C is the actual frequency implied by C, given the 706 observations, which we denote as A. A is computed as $A = \frac{C}{706}$. If the VAR calculation method is accurate, then A, the "actual" α, should equal the expected α.

Table 16

The results indicate that in two of the three cases, the A for the gamma-GARCH model is closest to the expected α level. The gamma-WTN model performs slightly better than the gamma-GARCH model at the 1% level, but this may be due to the low number of observations in this category.

All of the HOM models perform poorly. In all cases, either or both of the WTN and GARCH models more accurately predict the VAR. This echoes the efficiency regression results. All of the gamma models perform better than the corresponding delta models, usually by wide margins. For example, at $\alpha = 5\%$, the best of the delta models (delta-GARCH) achieves $A = 1.8\%$. But this performance is very poor compared to the corresponding gamma-GARCH method, which achieves $A = 4.7\%$.

7 Conclusion

Our results indicate that the existing first-order (delta) models favored by practitioners perform very poorly in many situations when the portfolio has gamma risk, and that usage of second-order models, such as our own, can significantly improve the practice of market risk measurement. For example, at a confidence level of $\alpha = 1\%$, the average absolute errors in predicting the VAR for the delta and gamma methods were 38% and 6%, respectively. Such models should continue to be used with caution, however, because even gamma models perform poorly in certain situations, such as when the option is essentially valueless.

Although the strongest case could be made for the GARCH model among the three state variable models considered, our results indicate that even the GARCH model leaves something to be desired. No model was unambiguously best under either an R^* or RMSPE criterion. This result indicates that other models should be implemented and tested. In particular, we suggest more research on IGARCH and EGARCH models. Moreover, it would be interesting to extend this analysis to other types of state variables and securities.

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Table 1: Put Approximation Error, $\alpha = 10\%$

The table below reports measurement errors when first-order and second-order approximations are used to calculate the VAR at $\alpha = 0.10$. The forecast time period for computing the VAR is 1 week. One European put option on a non-dividend paying stock is used as the control asset and is priced using the Black-Scholes formula. The variance of the stock is assumed to be known with certainty at $\sigma^2 = 0.04$. Each row shows VARs calculated for one combination of option time-to-maturity (τ), and option 'moneyness' ($\frac{S}{K}$). Each row shows the Black-Scholes option premium (P), key hedge ratios (P_t, P_S, P_{SS}, theta, delta, and gamma), the true VAR (ΔP), the VAR calculated using a first-order approximation to the portfolio function and the true distribution of security returns (ΔP_1), and the VAR calculated using a second-order approximation to the portfolio function and the true distribution of security returns, (ΔP_2). For all combinations of τ and $\frac{S}{K}$, the initial option position is normalized to $1,000. The values are computed using simulation as described in the text. Standard errors were all less than 1 percent of the estimates and are therefore omitted. The last two columns show the relative error (expressed in percent) of the two approximations to the true VAR. The three final rows show the mean percentage error (MPE), mean absolute percentage error (MAPE), and the root mean square percentage error (RMSPE).

	$\frac{S}{K}$	100*P	P_t	P_S	P_{SS}	ΔP	ΔP_1	ΔP_2	E_1	E_2
1.00	0.91	2.1	0.0	-0.2	1.3	-282	-322	-280	-14	1
1.00	0.95	3.2	0.0	-0.2	1.6	-258	-289	-255	-12	1
1.00	1.00	4.8	0.0	-0.3	1.8	-233	-256	-229	-10	1
1.00	1.05	6.9	0.0	-0.4	2.0	-209	-226	-205	-8	2
1.00	1.10	9.4	0.0	-0.5	2.0	-187	-200	-184	-7	2
0.50	0.91	1.3	0.0	-0.2	1.7	-410	-498	-411	-22	0
0.50	0.95	2.3	0.0	-0.3	2.3	-364	-427	-360	-17	1
0.50	1.00	4.0	0.0	-0.4	2.7	-317	-360	-311	-13	2
0.50	1.05	6.4	0.0	-0.5	2.8	-273	-300	-267	-10	2
0.50	1.10	9.4	0.0	-0.6	2.6	-234	-251	-229	-7	2
0.25	0.91	0.6	0.0	-0.1	2.0	-595	-818	-627	-37	-5
0.25	0.95	1.5	0.0	-0.2	3.1	-516	-656	-519	-27	0
0.25	1.00	3.1	0.0	-0.4	3.9	-433	-512	-422	-18	2
0.25	1.05	5.8	0.0	-0.6	3.9	-353	-393	-340	-11	3
0.25	1.10	9.4	0.0	-0.8	3.1	-284	-302	-274	-7	3
0.10	0.91	0.1	0.0	0.0	1.6	-893	-1782	-1304	-99	-46
0.10	0.95	0.6	-0.1	-0.2	4.2	-793	-1257	-874	-58	-10
0.10	1.00	2.2	-0.1	-0.4	6.2	-646	-833	-623	-29	4
0.10	1.05	5.3	0.0	-0.7	5.2	-480	-535	-447	-12	7
0.10	1.10	9.5	0.0	-0.9	2.5	-341	-352	-325	-3	4
MPE									-21	-1
MAPE									21	5
RMSPE									31	11

Table 2: Put Approximation Error, $\alpha = 5\%$

The table below reports measurement errors when first-order and second-order approximations are used to calculate the VAR at $\alpha = 0.05$. The forecast time period for computing the VAR is 1 week. One European put option on a non-dividend paying stock is used as the control asset and is priced using the Black-Scholes formula. The variance of the stock is assumed to be known with certainty at $\sigma^2 = 0.04$. Each row shows VARs calculated for one combination of option time-to-maturity (τ), and option 'moneyness' ($\frac{S}{K}$). Each row shows the Black-Scholes option premium (P), key hedge ratios (P_t, P_S, P_{SS}, theta, delta, and gamma), the true VAR (ΔP), the VAR calculated using a first-order approximation to the portfolio function and the true distribution of security returns (ΔP_1), and the VAR calculated using a second-order approximation to the portfolio function and the true distribution of security returns, (ΔP_2). For all combinations of τ and $\frac{S}{K}$, the initial option position is normalized to \$1,000. The values are computed using simulation as described in the text. Standard errors were all less than 1 percent of the estimates and are therefore omitted. The last two columns show the relative error (expressed in percent) of the two approximations to the true VAR. The three final rows show the mean percentage error (MPE), mean absolute percentage error (MAPE), and the root mean square percentage error (RMSPE).

τ	$\frac{S}{K}$	100*P	P_t	P_S	P_{SS}	ΔP	ΔP_1	ΔP_2	E_1	E_2
1.00	0.91	2.1	0.0	-0.2	1.3	-347	-411	-341	-19	2
1.00	0.95	3.2	0.0	-0.2	1.6	-319	-370	-313	-16	2
1.00	1.00	4.8	0.0	-0.3	1.8	-289	-329	-284	-14	2
1.00	1.05	6.9	0.0	-0.4	2.0	-261	-291	-257	-11	2
1.00	1.10	9.4	0.0	-0.5	2.0	-236	-258	-232	-9	2
0.50	0.91	1.3	0.0	-0.2	1.7	-490	-632	-485	-29	1
0.50	0.95	2.3	0.0	-0.3	2.3	-441	-543	-432	-23	2
0.50	1.00	4.0	0.0	-0.4	2.7	-389	-459	-380	-18	2
0.50	1.05	6.4	0.0	-0.5	2.8	-339	-385	-331	-14	2
0.50	1.10	9.4	0.0	-0.6	2.6	-293	-322	-287	-10	2
0.25	0.91	0.6	0.0	-0.1	2.0	-682	-1024	-695	-50	-2
0.25	0.95	1.5	0.0	-0.2	3.1	-606	-828	-598	-37	1
0.25	1.00	3.1	0.0	-0.4	3.9	-520	-650	-504	-25	3
0.25	1.05	5.8	0.0	-0.6	3.9	-433	-502	-419	-16	3
0.25	1.10	9.4	0.0	-0.8	3.1	-355	-389	-345	-9	3
0.10	0.91	0.1	0.0	0.0	1.6	-938	-2174	-1338	-132	-43
0.10	0.95	0.6	-0.1	-0.2	4.2	-865	-1556	-914	-80	-6
0.10	1.00	2.2	-0.1	-0.4	6.2	-741	-1049	-707	-42	5
0.10	1.05	5.3	0.0	-0.7	5.2	-581	-684	-546	-18	6
0.10	1.10	9.5	0.0	-0.9	2.5	-429	-454	-413	-6	4
MPE									-29	0
MAPE									29	5
RMSPE									41	10

Table 3: Put Approximation Error, $\alpha = 1\%$

The table below reports measurement errors when first-order and second-order approximations are used to calculate the VAR at $\alpha = 0.01$. The forecast time period for computing the VAR is 1 week. One European put option on a non-dividend paying stock is used as the control asset and is priced using the Black-Scholes formula. The variance of the stock is assumed to be known with certainty at $\sigma^2 = 0.04$. Each row shows VARS calculated for one combination of option time-to-maturity (τ), and option 'moneyness' ($\frac{S}{K}$). Each row shows the Black-Scholes option premium (P), key hedge ratios (P_t, P_S, P_{SS}, theta, delta, and gamma), the true VAR (ΔP), the VAR calculated using a first-order approximation to the portfolio function and the true distribution of security returns (ΔP_1), and the VAR calculated using a second-order approximation to the portfolio function and the true distribution of security returns, (ΔP_2). For all combinations of τ and $\frac{S}{K}$, the initial option position is normalized to $1,000. The values are computed using simulation as described in the text. Standard errors were all less than 1 percent of the estimates and are therefore omitted. The last two columns show the relative error (expressed in percent) of the two approximations to the true VAR. The three final rows show the mean percentage error (MPE), mean absolute percentage error (MAPE), and the root mean square percentage error (RMSPE).

τ	$\frac{S}{K}$	100*P	P_t	P_S	P_{SS}	ΔP	ΔP_1	ΔP_2	E_2	E_2
1.00	0.91	2.1	0.0	-0.2	1.3	-456	-582	-440	-28	4
1.00	0.95	3.2	0.0	-0.2	1.6	-423	-524	-410	-24	3
1.00	1.00	4.8	0.0	-0.3	1.8	-389	-468	-378	-20	3
1.00	1.05	6.9	0.0	-0.4	2.0	-354	-415	-346	-17	2
1.00	1.10	9.4	0.0	-0.5	2.0	-322	-368	-316	-14	2
0.50	0.91	1.3	0.0	-0.2	1.7	-618	-887	-584	-44	5
0.50	0.95	2.3	0.0	-0.3	2.3	-566	-766	-541	-35	5
0.50	1.00	4.0	0.0	-0.4	2.7	-510	-651	-491	-28	4
0.50	1.05	6.4	0.0	-0.5	2.8	-452	-547	-440	-21	3
0.50	1.10	9.4	0.0	-0.6	2.6	-399	-460	-390	-15	2
0.25	0.91	0.6	0.0	-0.1	2.0	-805	-1420	-734	-77	9
0.25	0.95	1.5	0.0	-0.2	3.1	-738	-1157	-683	-57	8
0.25	1.00	3.1	0.0	-0.4	3.9	-657	-916	-622	-39	5
0.25	1.05	5.8	0.0	-0.6	3.9	-568	-712	-548	-26	4
0.25	1.10	9.4	0.0	-0.8	3.1	-480	-555	-470	-16	2
0.10	0.91	0.1	0.0	0.0	1.6	-980	-2926	-1349	-199	zli-
0.10	0.95	0.6	-0.1	-0.2	4.2	-944	-2131	-930	-126	1
0.10	1.00	2.2	-0.1	-0.4	6.2	-866	-1462	-775	-69	10
0.10	1.05	5.3	0.0	-0.7	5.2	-736	-968	-704	-32	4
0.10	1.10	9.5	0.0	-0.9	2.5	-582	-651	-573	-12	2
MPE									-45	2
MAPE									45	6
RMSPE									63	10

Table 4: Call Approximation Error, $\alpha = 10\%$

The table below reports measurement errors when first-order and second-order approximations are used to calculate the VAR at $\alpha = 0.10$. The forecast time period for computing the VAR is 1 week. One European call option on a non-dividend paying stock is used as the control asset and is priced using the Black-Scholes formula. The variance of the stock is assumed to be known with certainty at $\sigma^2 = 0.04$. Each row shows VARS calculated for one combination of option time-to-maturity (τ), and option 'moneyness' ($\frac{S}{K}$). Each row shows the Black-Scholes option premium (P), key hedge ratios (P_t, P_S, P_{SS}, theta, delta, and gamma), the true VAR (ΔP), the VAR calculated using a first-order approximation to the portfolio function and the true distribution of security returns (ΔP_1), and the VAR calculated using a second-order approximation to the portfolio function and the true distribution of security returns, (ΔP_2). For all combinations of τ and $\frac{S}{K}$, the initial option position is normalized to $1,000. The values are computed using simulation as described in the text. Standard errors were all less than 1 percent of the estimates and are therefore omitted. The last two columns show the relative error (expressed in percent) of the two approximations to the true VAR. The three final rows show the mean percentage error (MPE), mean absolute percentage error (MAPE), and the root mean square percentage error (RMSPE).

τ	$\frac{S}{K}$	100*P	P_t	P_S	P_{SS}	ΔP	ΔP_1	ΔP_2	E_1	E_2
1.00	0.91	17.3	-0.1	0.8	1.3	-164	-168	-164	-3	0
1.00	0.95	14.4	-0.1	0.8	1.6	-181	-187	-181	-4	0
1.00	1.00	11.5	-0.1	0.7	1.8	-200	-210	-201	-5	-1
1.00	1.05	9.0	-0.1	0.6	2.0	-221	-235	-223	-6	-1
1.00	1.10	6.9	-0.1	0.5	2.0	-242	-261	-246	-8	-2
0.50	0.91	13.5	-0.1	0.8	1.7	-215	-222	-214	-4	0
0.50	0.95	10.4	-0.1	0.7	2.3	-247	-261	-248	-5	0
0.50	1.00	7.4	-0.1	0.6	2.7	-285	-308	-288	-8	-1
0.50	1.05	5.0	-0.1	0.5	2.8	-327	-364	-335	-11	-2
0.50	1.10	3.2	-0.1	0.4	2.6	-370	-424	-384	-15	-4
0.25	0.91	11.3	-0.1	0.9	2.0	-270	-280	-268	-4	1
0.25	0.95	7.9	-0.1	0.8	3.1	-330	-353	-329	-7	0
0.25	1.00	4.9	-0.1	0.6	3.9	-403	-453	-410	-13	-2
0.25	1.05	2.7	-0.1	0.4	3.9	-482	-578	-508	-20	-5
0.25	1.10	1.3	-0.1	0.2	3.1	-559	-721	-623	-29	-11
0.10	0.91	9.9	-0.1	1.0	1.6	-337	-344	-330	-2	2
0.10	0.95	6.1	-0.1	0.8	4.2	-462	-500	-451	-8	2
0.10	1.00	2.9	-0.2	0.6	6.2	-620	-762	-640	-23	-3
0.10	1.05	1.0	-0.1	0.3	5.2	-770	-1145	-939	-49	-22
0.10	1.10	0.2	-0.1	0.1	2.5	-877	-1630	-1397	-86	-59
MPE									-15	-5
MAPE									15	6
RMSPE									25	14

Table 5: Call Approximation Error, $\alpha = 5\%$

The table below reports measurement errors when first-order and second-order approximations are used to calculate the VAR at $\alpha = 0.05$. The forecast time period for computing the VAR is 1 week. One European call option on a non-dividend paying stock is used as the control asset and is priced using the Black-Scholes" formula. The variance of the stock is assumed to be known with certainty at $\sigma^2 = 0.04$. Each row shows VARs calculated for one combination of option time-to-maturity (τ), and option 'moneyness' ($\frac{S}{K}$). Each row shows the Black-Scholes option premium (P), key hedge ratios (P_t, P_S, P_{SS}, theta, delta, and gamma), the true VAR (ΔP), the VAR calculated using a first-order approximation to the portfolio function and the true distribution of security returns (ΔP_1), and the VAR calculated using a second-order approximation to the portfolio function and the true distribution of security returns, (ΔP_2). For all combinations of τ and $\frac{S}{K}$, the initial option position is normalized to $1,000. The values are computed using simulation as described in the text. Standard errors were all less than 1 percent of the estimates and are therefore omitted. The last two columns show the relative error (expressed in percent) of the two approximations to the true VAR. The three final rows show the mean percentage error (MPE), mean absolute percentage error (MAPE), and the root mean square percentage error (RMSPE).

τ	$\frac{S}{K}$	100*P	P_t	P_S	P_{SS}	ΔP	ΔP_1	ΔP_2	E_1	E_2
1.00	0.91	17.3	-0.1	0.8	1.3	-207	-215	-208	-4	0
1.00	0.95	14.4	-0.1	0.8	1.6	-227	-239	-228	-5	0
1.00	1.00	11.5	-0.1	0.7	1.8	-251	-267	-253	-6	-1
1.00	1.05	9.0	-0.1	0.6	2.0	-275	-298	-279	-8	-1
1.00	1.10	6.9	-0.1	0.5	2.0	-300	-331	-305	-10	-2
0.50	0.91	13.5	-0.1	0.8	1.7	-270	-283	-270	-5	0
0.50	0.95	10.4	-0.1	0.7	2.3	-308	-331	-310	-7	-1
0.50	1.00	7.4	-0.1	0.6	2.7	-353	-390	-357	-11	-1
0.50	1.05	5.0	-0.1	0.5	2.8	-400	-459	-410	-15	-2
0.50	1.10	3.2	-0.1	0.4	2.6	-447	-533	-465	-19	-4
0.25	0.91	11.3	-0.1	0.9	2.0	-338	-356	-337	-5	0
0.25	0.95	7.9	-0.1	0.8	3.1	-407	-448	-409	-10	0
0.25	1.00	4.9	-0.1	0.6	3.9	-488	-571	-498	-17	-2
0.25	1.05	2.7	-0.1	0.4	3.9	-571	-724	-603	-27	-6
0.25	1.10	1.3	-0.1	0.2	3.1	-648	-897	-720	-38	-11
0.10	0.91	9.9	-0.1	1.0	1.6	-423	-438	-417	-4	1
0.10	0.95	6.1	-0.1	0.8	4.2	-561	-632	-555	-13	1
0.10	1.00	2.9	-0.2	0.6	6.2	-717	-951	-749	-33	-4
0.10	1.05	1.0	-0.1	0.3	5.2	-847	-1407	-1028	-66	-21
0.10	1.10	0.2	-0.1	0.1	2.5	-927	-1975	-1458	-113	-57
MPE									-21	-6
MAPE									21	6
RMSPE									33	14

Table 6: Call Approximation Error, $\alpha = 1\%$

The table below reports measurement errors when first-order and second-order approximations are used to calculate the VAR at $\alpha = 0.01$. The forecast time period for computing the VAR is 1 week. One European call option on a non-dividend paying stock is used as the control asset and is priced using the Black-Scholes formula. The variance of the stock is assumed to be known with certainty at $\sigma^2 = 0.04$. Each row shows VARs calculated for one combination of option time-to-maturity (τ), and option 'moneyness' ($\frac{S}{K}$). Each row shows the Black-Scholes option premium (P), key hedge ratios (P_t, P_S, P_{SS}, theta, delta, and gamma), the true VAR (ΔP), the VAR calculated using a first-order approximation to the portfolio function and the true distribution of security returns (ΔP_1), and the VAR calculated using a second-order approximation to the portfolio function and the true distribution of security returns, (ΔP_2). For all combinations of τ and $\frac{S}{K}$, the initial option position is normalized to \$1,000. The values are computed using simulation as described in the text. Standard errors were all less than 1 percent of the estimates and are therefore omitted. The last two columns show the relative error (expressed in percent) of the two approximations to the true VAR. The three final rows show the mean percentage error (MPE), mean absolute percentage error (MAPE), and the root mean square percentage error (RMSPE).

τ	$\frac{S}{K}$	100*P	P_t	P_S	P_{SS}	ΔP	ΔP_1	ΔP_2	E_1	E_2
1.00	0.91	17.3	-0.1	0.8	1.3	-285	-301	-287	-6	-1
1.00	0.95	14.4	-0.1	0.8	1.6	-311	-334	-314	-7	-1
1.00	1.00	11.5	-0.1	0.7	1.8	-340	-373	-344	-10	-1
1.00	1.05	9.0	-0.1	0.6	2.0	-371	-416	-376	-12	-1
1.00	1.10	6.9	-0.1	0.5	2.0	-401	-461	-409	-15	-2
0.50	0.91	13.5	-0.1	0.8	1.7	-368	-396	-371	-8	-1
0.50	0.95	10.4	-0.1	0.7	2.3	-415	-462	-420	-11	-1
0.50	1.00	7.4	-0.1	0.6	2.7	-468	-544	-476	-16	-2
0.50	1.05	5.0	-0.1	0.5	2.8	-522	-637	-536	-22	-3
0.50	1.10	3.2	-0.1	0.4	2.6	-573	-737	-594	-29	-4
0.25	0.91	11.3	-0.1	0.9	2.0	-458	-498	-461	-9	-1
0.25	0.95	7.9	-0.1	0.8	3.1	-537	-624	-546	-16	-2
0.25	1.00	4.9	-0.1	0.6	3.9	-624	-791	-643	-27	-3
0.25	1.05	2.7	-0.1	0.4	3.9	-706	-996	-743	-41	-5
0.25	1.10	1.3	-0.1	0.2	3.1	-775	-1224	-840	-58	-8
0.10	0.91	9.9	-0.1	1.0	1.6	-571	-614	-574	-8	-1
0.10	0.95	6.1	-0.1	0.8	4.2	-715	-878	-731	-23	-2
0.10	1.00	2.9	-0.2	0.6	6.2	-848	-1303	-896	-54	-6
0.10	1.05	1.0	-0.1	0.3	5.2	-934	-1896	-1076	-103	-15
0.10	1.10	0.2	-0.1	0.1	2.5	-975	-2617	-1483	-168	-52
MPE									-32	-6
MAPE									32	6
RMSPE									50	13

Table 7: Summary of Approximation Error (%)

The table below summarizes the simulation results in Tables 1 through 6. Three different metrics, mean percentage error (MPE), mean absolute percentage error (MAPE), and root mean square percentage error (RMSPE), are presented for puts and calls at three different α levels, 10%, 5%, and 1%.

Option Type	α level	MPE ΔP_1	MPE ΔP_2	MAPE ΔP_1	MAPE ΔP_2	RMSPE ΔP_1	RMSPE ΔP_2
	10%	-21	-1	21	5	31	11
Puts	5	-29	0	29	5	41	10
	1	-45	2	45	6	63	10
	10	-15	-5	15	6	25	14
calls	5	-21	-6	21	6	33	14
	1	-32	-6	32	6	50	13

Table 8: Data Sample of Weekly Equity Closing Prices

The price series below are excerpted from the twenty-six years (1969 through 1994) used in the study. The data was sampled weekly (on successive Wednesdays) yielding an average of 1356 observations. The Equity prices are from the CRSP database.

Date	Dow Chemical	Exxon	Union Carbide	Coca-Cola	S&P 500
10/05/94	73.125	56.875	32.625	48.625	453.520
10/12/94	74.375	58.625	32.125	49.875	465.470
10/19/94	75.250	59.500	33.500	50.625	470.280
10/26/94	73,125	61.250	34.125	50.000	462.610
11/02/94	69.750	62.000	31.000	50.250	466.510
11/09/94	69.000	60.250	31.000	50.625	465.420
11/16/94	67.625	60.625	30.875	52.250	465.620
11/23/94	61.625	60.500	27.000	51.750	449.930
11/30/94	64.000	60.375	28.625	51.125	453.690
12/07/94	64.375	60.750	28.750	50.875	451.230
12/14/94	66.875	61.000	30.125	50.875	454.970
12/21/94	67.500	61.500	30.000	52.750	459.610
12/28/94	67.500	61.750	29.750	52.250	460.860

Table 9: Illustrative Σ^1, Weekly State Variable Covariance Matrix Under X-1

Regression results appear below for the following model: $\tilde{\mathbf{r}}_t = \mu + \tilde{\epsilon}, \quad \tilde{\epsilon} \sim N_n(\mathbf{0}, \mathbf{\Sigma^{1r}})$ The raw data series are as described in Table 8. The correlations and volatilities (σ) are computed using OLS on the returns from the CRSP database. The values are as of 12/28/94. $\mathbf{\Sigma^{1r}}$ is computed by combining the vector of volatilities (σ, $n \times 1$) and the matrix of correlations ($\mathbf{R}$, $n \times n$) in the usual way, i.e., $\mathbf{\Sigma^{1r}} = \sigma'\mathbf{R}\sigma$. All values were highly significant.

	R				
	DOW	EXX	UNCA	COKE	S&P 500
DOW	1.000	0.335	0.555	0.266	0.585
EXX		1.000	0.256	0.181	0.439
UNCA			1.000	0.158	0.469
COKE				1.000	0.605
S&P 500					1.000
σ	0.0377	0.0250	0.0453	0.0406	0.0247
μ	0.0040	0.0038	0.0053	0.0092	0.0042

Table 10: Illustrative Σ_t^{2r}, Weekly State Variable Covariance Matrix Under X-2

Calculation results appear below for the following model: $\sigma_{ij(t)} = \sum_{k=0}^{\infty} \omega_k (r_{i(t-k)} - \mu_{r_i})(r_{j(t-k)} - \mu_{r_j})$ where $\omega_k = \lambda^k(1-\lambda)$, and λ is 0.95. The raw data series are as described in Table 8. The values are as of 12/28/94. $\mathbf{\Sigma^{3r}}$ is computed by combining the vector of volatilities (σ, $n \times 1$) and the matrix of correlations ($\mathbf{R}$, $n \times n$) in the usual way, i.e., $\mathbf{\Sigma^{3r}} = \sigma'\mathbf{R}\sigma$.

	R				
	DOW	EXX	UNCA	COKE	S&P 500
DOW	1.000	0.649	0.708	0.423	0.734
EXX		1.000	0.524	0.333	0.655
UNCA			1.000	0.309	0.627
COKE				1.000	0.674
S&P 500					1.000
σ	0.0363	0.0167	0.0518	0.0276	0.0166
μ	0.0040	0.0038	0.0053	0.0092	0.0042

Table 11: Illustrative Σ_t^{3r}, Weekly State Variable Covariance Matrix Under X-3

Numerical maximization results appear below for the following model:

$$\begin{aligned} \tilde{\mathbf{r}}_t &= \mu + \mathbf{u}_t \\ \mathbf{u}_t|\Psi_{t-1} &\sim N(0, \Sigma_t^{3r}) \end{aligned} \tag{1}$$

where $\tilde{\mathbf{r}}, \mu, \mathbf{u}_t$ are $n \times 1$ and Σ_t^{3r} is $n \times n$. Ψ_{t-1} is the information set at $t-1$. Individual elements of Σ_t^{3r} are given as follows:

$$\begin{aligned} v_{ii(t)} &= \alpha_{0i} + \alpha_{1i} u_{i,t-1}^2 + \beta_i v_{ii(t-1)} \\ \text{and} \quad v_{ij(t)} &= \rho_{ij} \sqrt{v_{ii(t)}} \sqrt{v_{jj(t)}} \end{aligned} \tag{2}$$

The covariance matrix can be written as $\Sigma_t^{3r} = \mathbf{V_t}\, \mathbf{R}\, \mathbf{V_t}$ where $\mathbf{V_t}$ is a time-varying diagonal matrix with typical element is $\sqrt{v_{ii(t)}}$, and $\mathbf{R}$ is a time-invariant matrix with typical element ρ_{ij}. The raw data series are as described in Table 8. The values are as of 12/28/94. Σ_t^{3r} is computed by combining the matrix of volatilities ($\mathbf{V_t}$, $n \times 1$) and the matrix of correlations ($\mathbf{R}$, $n \times n$) in the usual way, i.e., $\Sigma_t^{3r} = \mathbf{V_t R V_t}$. The means of the return series were 0.0045,0.0036, 0.0061, 0.0096, 0.0040 with standard errors 0.0013, 0.0010, 0.0018, 0.0015, 0.0009, respectively. Here are the numerical maximization results:

$$\begin{aligned}
\text{DOW:} \quad v_t &= \underset{(0.000045)}{0.000108} + \underset{(0.025)}{0.067} u_{t-1}^2 + \underset{(0.053)}{0.851} v_{t-1} \\
\text{EXX:} \quad v_t &= \underset{(0.000012)}{0.000025} + \underset{(0.014)}{0.0438} u_{t-1}^2 + \underset{(0.029)}{0.914} v_{t-1} \\
\text{UNCA:} \quad v_t &= \underset{(0.000118)}{0.000447} + \underset{(0.021)}{0.101} u_{t-1}^2 + \underset{(0.069)}{0.674} v_{t-1} \\
\text{COKE:} \quad v_t &= \underset{(0.000046)}{0.000121} + \underset{(0.020)}{0.100} u_{t-1}^2 + \underset{(0.041)}{0.823} v_{t-1} \\
\text{S\&P 500:} \quad v_t &= \underset{(0.000014)}{0.000057} + \underset{(0.022)}{0.100} u_{t-1}^2 + \underset{(0.040)}{0.794} v_{t-1}
\end{aligned} \tag{3}$$

The constant correlation matrix is:

$$R = \begin{bmatrix} 1.000 & \underset{(0.035)}{0.296} & \underset{(0.029)}{0.520} & \underset{(0.038)}{0.220} & \underset{(0.036)}{0.271} \\ & 1.000 & \underset{(0.040)}{0.171} & \underset{(0.040)}{0.171} & \underset{(0.025)}{0.578} \\ & & 1.000 & \underset{(0.030)}{0.412} & \underset{(0.032)}{0.456} \\ & & & 1.000 & \underset{(0.029)}{0.585} \\ & & & & 1.000 \end{bmatrix}$$

When this model is applied to the return series as' of 12/28/94, the predicted return volatilities for the following week ($\sqrt{v_{t+1}}$) are 0.0375, 0.0200, 0.0452, 0.0347, 0.0207.

Table 12: Root Mean Square Prediction Error (RMSPE) for X-1, X-2, and X-3

The three state variable return models (normal, X-1; weighted-normal, X-2; and GARCH, X-3) were applied as described in the text.

	DOW		EXX		UNCA		COKE		S&P 500	
Model	Rank	RMSPE	Rank	RMSPE	Rank	RMSPE	Rank	RMSPE	Rank	RMSPE
X-1	3	0.00302	3	0.00111	3	0.00433	3	0.00379	3	0.00158
X-2	1	0.00297	1	0.00110	1	0.00428	2	0.00378	2	0.00155
X-3	1	0.00297	1	0.00110	2	0.00431	1	0.00375	1	0.00154

Table 13: Regression Tests of Efficiency for X-1, X-2, and X-3

The results of efficiency regressions $r_t^2 = b_0 + b_1\hat{\sigma}_t^2 + \epsilon_t$ for each of the three models (X-1, X-2, and X-3) for all five state variables appear below. Newey and West (1987) autocorrelation- and heteroskedasticity-consistent standard errors are in parentheses below the estimated parameters. The $\chi^2(2)$ statistic is presented for the joint hypothesis $H_0 : b_0 = 0, b_1 = 1$ with p-value below. The superscripts "@Y' and "*" indicate significance of the durbin-watson (DW) test statistic at 5% and 1% levels, respectively.

Model for $\hat{\sigma}_t^2$	b_0	b_1	$\chi^2(2)$	DW	R^2
		DOW			
X-1 - HOM	00073	8	889	146	0007
	(0:0025)	0.012		0.012	
X-2 - WTN	0.0004	0.70	5.23	1.57	0.039
	(0.0002)	(0.20)	0.071		
X-3 - GARCH	-0.0005	1.35	2.020	1.59	0.037
	(0.0005)	(0.37)	0.366		
		EXX			
X-1 - HOM	-0.0012	2.37	17.38	1.75*	0.016
	(0.0005)	(0.63)	0.000		
X-2 - WTN	0.0002	0.70	4.20	1.84°	0.034
	(0.0001)	(0.16)	0.120		
X-3 - GARCH	-0.0002	1.15	4.81	1.92°	0.034
	(0.0004)	(0.20)	0.088		
		UNCA			
X-1 - HOM	0.0022	-0.14	5.12	1.50	0.007
	(0.0011)	(0.64)	0.075		
X-2 - WTN	0.0007	0.65	4.98	1.60	0.029
	(0.0003)	(0.22)	0.081		
X-3 - GARCH	0.0010	0.52	6.78	1.59	0.016
	(0.0004)	(0.21)	0.032		
		COKE			
X-1 - HOM	0.0105	-5.05	3.90	1.64	0.012
	(0.0069)	(3.96)	0.139		
X-2 - WTN	0.0008	0.55	11.41	1.70*	0.016
	(0.0003)	(0.20)	0.003		
X-3 - GARCH	'0.0006'	'0.68'	4.45	1.73"	0.018
	(0.0003)	(0.22)	0.106		
		S&P 500			
X-1 - HOM	0.0036	-4.36	4.37	1.70*	0.006
	(0.0032)	(4.75)	0.110		
X-2 - WTN	0.0002	0.71	6.57	1.83°	0.038
	(0.0001)	(0.17)	0.036		
X-3 - GARCH	0.0001	1.11	0.79	1.94°	0.048
	(0.0001)	(0.23)	0.678		

Table 14: Inventory of Securities by State Variable

Five of the securities in the portfolio are standard European options on each of the five state variables. The option pricing parameters are purely hypothetical. The Black-Scholes pricing model was used to value all options. X is the value of the underlying state variable; K is the strike price; r_f is the constant risk free interest rate; τ is the time to expiration in years; σ is the annualized volatility of the underlying; The number of contracts was computed to achieve a uniform total value per state variable of $200.00.

State Variable	Option Type	Option Pricing Parameters X/K	r_f	σ	τ
DOW	Put	1.05	7%	20.0%	1.00
EXX	Call	1/1.05	7	20.0	0.25
UNCA	Put	1.05	7	20.0	0.50
COKE	Call	1/1.05	7	20.0	1.00
S&P 500	Put	1.05	7	20.0	0.25

Table 15: VAR Illustration

Moments and other statistics such as the VAR are shown below for the six VAR calculation methods (consisting of all combinations of the two portfolio approximation methods and the three state variable models) for one calculation date, 12/21/94. We provide six statistics: the mean (μ), the variance (σ^2), the skewness, the excess kurtosis, and the first, fifth, and tenth percentiles. The latter three of these statistics are the VARS at 1%, 5%, and 10%, respectively, Omitted entries are zero.

Portfolio Approximation	State Variable Model	μ	σ	Skewness	Excess Kurtosis	(VAR @ 10%)	(VAR @ 5%)	(VAR @ 1%)
P-1 – delta	X-1 – HOM	-52.2	147.1			-240.7	-294.2	-394.4
P-1 – delta	X-2 – WTN	-52.2	149.1			-243.2	-297.4	-399.4
P-1 – delta	X-3 – GARCH	-52.2	129.1			-217.6	-264.5	-252.5
P-2 – gamma	X-1 – HOM	-9.5	152.1	0.66	0.76	-189.2	-227.5	-291.6
P-2 – gamma	X-2 – WTN	-13.8	154.7	0.93	1.34	-189.2	-220.6	-265.8
P-2 – gamma	X-3 – GARCH	-15.6	133.7	0.79	1.04	-170.7	-201.0	-249.3

Table 16: Out-of-Sample VAR Comparison

The results below compare the performance of the six VAR calculation methods at correctly predicting the VAR. The methods were applied to a hypothetical portfolio on an out-of-sample basis for 706 weeks from 1981 through 1994. The number of weeks actual portfolio losses exceeded the VAR (C) is shown for each of three different α levels. Next to C is the implied actual frequency A, computed as $A = \frac{C}{706}$.

Portfolio Fen. Approximation Model	State Variable Model	Expected α Level 10% C	A	5% C	A	1% C	A
P-1 - delta	X-1 - HOM	27	3.8%	12	1.7%	o	0.0%
P-1 - delta	X-2 - WTN	29	4.1	10	1.4	1	0.1
P-1 - delta	X-3 - GARCH	32	4.5	13	1.8	3	0.4
P-2 - gamma	X-1 - HOM	51	7.2	27	3.8	4	0.6
P-2 - gamma	X-2 - WTN	53	7.5	29	4.1	7	1.0
P-2 - gamma	X-3 - GARCH	60	8.5	33	4.7	10	1.4